Unstable Particles and Non-Conserved Currents: A Generalization of the Fermion-Loop Scheme

Giampiero Passarino

Dipartimento di Fisica Teorica, Università di Torino, Italy
INFN, Sezione di Torino, Italy

Abstract

The incorporation of finite-width effects in the theoretical predictions for tree-level processes $e^+e^- \to n$ fermions requires that gauge invariance must not be violated. Among various schemes proposed in the literature, the most satisfactory, from the point of view of field theory is the so-called Fermion-Loop scheme. It consists in the re-summation of the fermionic one-loop corrections to the vector-boson propagators and the inclusion of all remaining fermionic one-loop corrections, in particular those to the Yang–Mills vertices. In the original formulation, the Fermion-Loop scheme requires that vector bosons couple to conserved currents, i.e., that the masses of all external fermions be neglected. There are several examples where fermion masses must be kept to obtain a reliable prediction. The most famous one is the so-called single-W production mechanism, the process $e^+e^- \to e^-\overline{\nu}_e f_1 \overline{f}_2$ where the outgoing electron is collinear, within a small cone, with the incoming electron. Therefore, m_e cannot be neglected. Furthermore, among the 20 Feynman diagrams that contribute (for $e\overline{\nu}_e u\overline{d}$ final states, up to 56 for $e^+e^-\nu_e\overline{\nu}_e$) there are multi-peripheral ones that require a non-vanishing mass also for the other fermions. A generalization of the Fermion-Loop scheme is introduced to account for external, non-conserved, currents. Dyson re-summed transitions are introduced without neglecting the $p_\mu p_\nu$-terms and including the contributions from the Higgs-Kibble ghosts in the 't Hooft-Feynman gauge. Running vector boson masses are introduced and their relation with the corresponding complex poles are investigated. It is shown that any $\mathcal{S}$-matrix element takes a very simple form when written in terms of these running masses. A special example of Ward identity, the U(1) Ward identity for single-W, is derived in a situation where all currents are non-conserved and where the top quark mass is not neglected inside loops.

1 Introduction.

The incorporation of finite-width effects in the theoretical predictions for LEP2 processes and beyond necessitates a careful treatment. Independently of how finite widths of propagating particles are introduced, this requires a re-summation of the vacuum-polarization effects. Furthermore, the principle of gauge invariance must not be violated, i.e., the Ward identities, have to be preserved.

In a series of two papers, [1] and [2], one can find a complete description of several schemes that allow the incorporation of finite-width effects in tree-level amplitudes without spoiling gauge invariance.

In [1] it was argued that the preferable (Fermion-Loop) scheme consists in the re-summation of the fermionic one-loop corrections to the vector-boson propagators and the inclusion of all remaining fermionic one-loop corrections, in particular those to the Yang–Mills vertices. This re-summation of one-particle-irreducible (1PI) fermionic $\mathcal{O}(\alpha)$ corrections involves the closed set of all $\mathcal{O}\left([N_c^f \alpha/\pi]^i\right)$ (leading color-factor) corrections, and is as such manifestly gauge-invariant. These corrections constitute the bulk of the width effects for gauge bosons and an important part of the complete set of weak corrections.

In Ref. [1] the main incentive was the discussion of the process $e^+e^- \to e^-\overline{\nu}_e u\overline{d}$ at small scattering angles and LEP2 energies. Naive inclusion of the finite W-boson width breaks U(1) electromagnetic gauge invariance and leads to a totally wrong cross-section in the collinear limit, as e.g. discussed in Ref. [3]. By taking into account in addition the imaginary parts arising from cutting the massless fermion loops in the triple-gauge-boson vertex, U(1) gauge invariance is restored and a sensible cross-section is obtained.

In Ref. [2] the authors presented the details of the full-fledged Fermion-Loop scheme, taking into account the complete fermionic one-loop corrections including all real and imaginary parts, and all contributions of the massive top quark. A proper treatment of the neutral gauge-boson propagators is performed by solving the Dyson equations for the photon, Z-boson, and mixed photon–Z propagators. This is necessary to guarantee the unitarity cancellations at high energies. The top-quark contributions are particularly important for delayed-unitarity effects. In this respect also terms involving the totally-antisymmetric ε-tensor (originating from vertex corrections) are relevant. While such terms are absent for complete generations of massless fermions owing to the anomaly cancellations, they show up for finite fermion masses. As the ε-dependent terms satisfy the Ward identities by themselves, they can be left out in more minimal treatments like the one used in Ref. [1].

In Ref. [2] a renormalization of the fermion-loop corrections is formulated that uses the language of running couplings. One rewrites bare amplitudes in terms of these renormalized couplings and demonstrates that the resulting renormalized amplitudes respect gauge invariance, i.e., that they fulfill the relevant Ward identities.

In both papers the external fermionic currents are assumed to be conserved, i.e. one neglects the masses of the external fermions. The effect of including masses is notoriously very small, except in collinear regions like, for instance,

the one of single-W production.

In other words, one should remember that the current attached to the photon propagator is strictly conserved, while the ones attached to the (massive) vector boson propagators are not. The missing terms are of order m_f^2 and, therefore, they are negligible if we can show that collinear limits and high-energy limit are not upset by ignoring these terms. Although no formal proof of the extension of the Fermion-Loop scheme is found in the literature, some partial considerations and a numerical analysis are given in Ref. [4].

In Ref. [1], the CC20 process, $e^+ e^- \to e^- \overline{\nu}_e u \overline{d}$ has been analyzed in the so-called WW configuration, where we require that the outgoing electron is away from the collinear region, $\theta_e > \theta_c$, although numerical results were presented with θ_c as low as $0.1°$. However, the opposite limit, $\theta_e < \theta_c$, is also of theoretical and experimental importance, defining the so-called single-W production cross-section. A complete description of all aspects present in single-W production can be found in Ref [5]. Complementary aspects can also be found in Ref. [6].

This process has been extensively analyzed in the literature. It is a sensitive probe of anomalous electromagnetic couplings of the W boson and represents a background to searches for new physics beyond the standard model.

As we have discussed above, the issue of gauge invariance in the CC20 family has been solved by the introduction of the Fermion-Loop scheme but several subtleties remain, connected with the region of vanishing scattering angle of the electron and with the limit of massless final state fermions in a fully extrapolated setup. A satisfactory solution to compute the total cross-section is, therefore, given by the extension of the Fermion-Loop scheme to the case of external, non-conserved currents.

First of all, for single$W$ production one cannot neglect the electron mass, nor in the matrix element, neither in the kinematics of the process. However, keeping a finite electron mass through the calculation is not enough. Massless quarks in the final state induce a singularity, even for finite m_e, if a cut is not imposed on the invariant mass $M(u\overline{d})$.

With a cut on $M(\overline{d}u)$ the singularity at zero momentum transfer ($Q^2 = 0$) is avoided but we still have additional singularities. Indeed, there are two multi-peripheral diagrams contributing to the CC20 process $e^+ e^- \to e^- \overline{\nu}_e f_1 \overline{f}_2$, see Fig. 1. When $Q^2 = 0$, i.e. the electron is lost in the beam pipe, and the (massless) $f_1(f_2)$-fermion is emitted parallel to the (quasi-real) photon then the internal fermion propagator will produce an enhancement in the cross section. Taking into account a $\ln m_e^2$ from the photon flux-function, three options follow:

1. to consider massive (f_1/f_2) fermions, giving a result that is proportional to $\ln m_e^2 \ln m_f^2$,

2. to use massless fermions, giving instead $\ln^2 m_e^2$,

3. to introduce an angular cut on the outgoing f_1 and $\overline{f}_2$ fermions with respect to the beam axis, $\theta(f_1, \bar{f}_2) \geq \theta_{\mathrm{cut}}$, giving $\ln m_e^2 \ln \theta_{\mathrm{cut}}$.

The first option is clean but ambiguous when the final state fermions are light quarks, what to use for m_u, m_d? The second one presents no problems for a fully leptonic CC20 final state but completely fails to describe quarks, as it can be shown by discussing QCD corrections [5]. The last option is also theoretically clean and can be used to give differential distributions for the final state jets. It is, however, disliked by the experimentalists when computing the total sample of events: hadronized jets are seen and not isolated quarks. Even if the quark is parallel to the beam axis the jet could be broad enough and the event selected. These events are also interesting since they correspond to a situation where the electron and one of the quarks are lost in the beam pipe, while the other quark is recoiling against the neutrino, i.e. one has a totally unbalanced mono-jet structure, background to new particle searches[1].

The singularity induced by massless quarks in $e^+ e^- \to e^- \overline{\nu}_e u \overline{d}$ can only be treated within the context of QCD final state corrections and of the photon hadronic structure function (PHSF) scenario. To discuss QCD corrections in the low $(u\overline{d})$ invariant mass region is not the purpose of this work, therefore we assume that the light quarks have a mass and that no kinematical cuts are imposed on the process. Our goal will be to formulate a consistent scheme that takes into account all fermion masses.

For many purposes it is enough to introduce a fixed width for the W, both for the s-channel and the t-channel, i.e. the fixed-width scheme. However, there is another important point in favor of adopting the full Fermion-Loop scheme, since it guarantees automatically the correct choice of scale for the running of α_{QED}. The latter is particularly relevant in a process that is dominated by a small momentum transfer.

In dealing with theoretical predictions we must distinguish between an Input-Parameter-Set (IPS) and a Renormalization-Scheme (RS). IPS one always has, RS comes only when one starts including loop corrections. The IPS can be made equal in all calculations, RS is the author's choice, very much as the choice of gauge.

Apart from some recent development, each calculation aimed to provide some estimate for $e^+ e^- \to 4\,\mathrm{f}$ production is, at least nominally, a tree level calculation. Among other things it will require the choice of some IPS and of certain relations among the parameters. In the literature, although improperly, this is usually referred to as the choice of the Renormalization Scheme.

Typically we have at our disposal four experimental data point (plus α_s), i.e. the measured vector boson masses M_z, M_w and the coupling constants, G_F and α. However we only have three bare parameters at our disposal, the charged vector boson mass, the $SU(2)$ coupling constant and the sinus of the weak mixing angle. While the inclusion of one loop corrections would allow us to fix at least the value of the top quark mass from a consistency relation, this cannot be done at the tree level. Thus, different choices of the basic relations among the input parameters can lead to different results with deviations which, in some case, can be sizeable.

---

[1] M. Grünewald, private communication.

4

For instance, a possible choice is to fix the coupling constant g as

$$g^2 = \frac{4\pi\alpha}{s_W^2}, \quad s_W^2 = \frac{\pi\alpha}{\sqrt{2}G_F M_W^2}, \tag{1}$$

where G_F is the Fermi coupling constant. Another possibility would be to use

$$g^2 = 4\sqrt{2}G_F M_W^2, \tag{2}$$

but, in both cases, we miss the correct running of the coupling. Ad hoc solutions should be avoided, and the running of the parameters must always follow from a fully consistent scheme. Therefore, the only satisfactory solution is in the extension of the full Fermion-Loop to having non zero external masses, or non-conserved currents. Unless, of course, one can compute the full set of corrections.

We will term massive-massless the version of the Fermion-Loop scheme developed in [2]. Our generalization will be denoted as the massive-massive version of the Fermion-Loop scheme. Work is in progress towards the implementation of the massive-massive Fermion-Loop scheme in the Fortran code `WTO` [7]. The implementation of the Imaginary–Fermion-Loop scheme (see Ref. [2] for a definition) has been done in the Fortran code `WPHACT` [8].

For a review on the status of single-W we refer to the work of Ref. [9] and to some recent activity within the LEP2/MC Workshop[2] with comparisons among different groups [10]. Experimental findings are reported in [11].

The outline of the paper will be as follows. In Sect. 2 we recall the building blocks for the construction of the massive-massless Fermion-Loop scheme. In Sect. 3 we give the explicit construction of all transitions in the charged sector of the theory. The running W mass is introduced in Sect. 4. The single-W Ward identity is proved in Sect. 5. With Sect. 6 we give a detailed discussion of the numerically relevant approximations in the massive-massive Fermion-Loop scheme. Cancellation of ultraviolet divergences, within the Fermion-Loop scheme, is examined in Sect. 7. Re-summation in the neutral sector of the theory is discussed in Sect. 8, where we also introduce the notion of Z running mass and describe its relation with the other running parameters of the scheme. In Sect. 9 the imaginary parts of all corrections are explicitly computed.

2 The Fermion-Loop scheme

In this Section we briefly discuss the main ingredients entering into the Fermion-Loop scheme [2]. Here, we assume that all external currents are conserved, i.e. we assume that the external world is massless.

In the 't Hooft–Feynman gauge, the $\delta_{\mu\nu}$ part of the vector–vector transitions can be cast in the following form [12], where $s_\theta(c_\theta)$ is the sine(cosine) of the weak mixing angle:

$$S_{\gamma\gamma} = \frac{g^2 s_\theta^2}{16\,\pi^2}\,\Pi_{\gamma\gamma}(p^2)\,p^2, \quad S_{zz} = \frac{g^2}{16\pi^2 c_\theta^2}\,\Sigma_{zz}(p^2),$$

$$S_{z\gamma} = \frac{g^2 s_\theta}{16\pi^2 c_\theta} \Sigma_{z\gamma}(p^2), \quad S_{WW} = \frac{g^2}{16\pi^2} \Sigma_{WW}(p^2). \tag{3}$$

Next we have can transform to the $(3, Q)$ basis, where:

$$\begin{aligned}
\Sigma_{zz}(p^2) &= \Sigma_{33}(p^2) - 2s_\theta^2 \Sigma_{3Q}(p^2) + s_\theta^4 \Pi_{\gamma\gamma}(p^2) p^2, \\
\Sigma_{z\gamma}(p^2) &= \Sigma_{3Q}(p^2) - s_\theta^2 \Pi_{\gamma\gamma}(p^2) p^2.
\end{aligned} \tag{4}$$

To recall the derivation of the Fermion-Loop scheme, we start with some another drastic approximation; namely, we put to zero also all fermions masses in loops, i.e. we assume a massless internal world. Under this assumption we will introduce the following quantities:

$$\begin{aligned}
B_n \equiv B_n\left(p^2; 0, 0\right) &= 2B_{21}\left(p^2; 0, 0\right) - B_0\left(p^2; 0, 0\right), \\
B_c \equiv B_c\left(p^2; 0, 0\right) &= B_{21}\left(p^2; 0, 0\right) + B_0\left(p^2; 0, 0\right) = \frac{1}{2} B_n.
\end{aligned} \tag{5}$$

Here the $B_{ij}\left(p^2; 0, 0\right)$ are scalar one-loop integrals [13]. Furthermore, we will use the fact that a p^2 can be factorized:

$$\Sigma_{3Q}(p^2) = \Pi_{3Q}(p^2) p^2, \qquad \Sigma_{33}(p^2) = \Pi_{33}(p^2) p^2. \tag{6}$$

As a result, all the vector boson–vector boson transitions simplify drastically:

$$\begin{aligned}
\Pi_{\gamma\gamma}(s) &= \frac{32}{3} \sum_g B_n, \quad \Pi_{3Q}(s) = 4 \sum_g B_n = \frac{3}{8} \Pi_{\gamma\gamma}(s), \\
\Sigma_{33}(s) &= -4s \sum_g B_n = -s\Pi_{3Q}(s), \\
\Sigma_{WW}(s) &= -8s \sum_g B_c = -4s \sum_g B_n = -s\Pi_{33}(s),
\end{aligned} \tag{7}$$

where $p^2 = -s$ and where the sum is over the fermion generations.

The resulting expressions retain some simplicity even if we do not ignore the top quark mass. Self-energies may still be written in compact form:

$$\begin{aligned}
\Pi_{\gamma\gamma}(s) &= \frac{32}{3} \sum_g B_n + \frac{16}{3} \overline{B}_n, \quad \Pi_{3Q}(s) = 4 \sum_g B_n + 2\overline{B}_n, \\
\Sigma_{33}(s) &= -s \Pi_{3Q}(s) + \frac{1}{2} s \overline{B}_n - \frac{3}{2} m_t^2 B_0\left(-s; m_t, m_t\right), \\
\Sigma_{WW}(s) &= -s\left[4 \sum_g B_n + 6\overline{B}_c\right] - 3 m_t^2 B_{mc},
\end{aligned} \tag{8}$$

where we have introduced new auxiliary functions, defined by

$$\begin{aligned}
\overline{B}_n &= B_n\left(-s; m_t, m_t\right) - B_n\left(-s; 0, 0\right), \\
\overline{B}_c &= B_c\left(-s; m_t, 0\right) - B_c\left(-s; 0, 0\right), \\
B_{mc} &= B_1\left(-s; m_t, 0\right) + B_0\left(-s; m_t, 0\right).
\end{aligned} \tag{9}$$

A useful way of presenting these results will be to split the universal part, proportional to $\Pi_{3Q}(s)$, from the remainder:

$$\Sigma_{33}(s) = -s\,\Pi_{3Q}(s) + f_z(s), \quad \Sigma_{WW}(s) = -s\,\Pi_{3Q}(s) + f_W(s), \qquad (10)$$

where the two f-functions are expressible as

$$
\begin{aligned}
f_W(s) &= -2\,s\,\Big[3B_{21}\left(-s;m_t,0\right) - 2B_{21}\left(-s;m_t,m_t\right) - B_{21}\left(-s;0,0\right) \\
&\quad +3B_1\left(-s;m_t,0\right) - 3B_1\left(-s;0,0\right) + B_0\left(-s;m_t,m_t\right) - B_0\left(-s;0,0\right)\Big] \\
&\quad -3\,m_t^2\,\Big[B_1\left(-s;m_t,0\right) + B_0\left(-s;m_t,0\right)\Big], \\
f_z(s) &= \frac{1}{2}\,s\,\Big[2B_{21}\left(-s;m_t,m_t\right) - 2B_{21}\left(-s;0,0\right) \\
&\quad - B_0\left(-s;m_t,m_t\right) + B_0\left(-s;0,0\right)\Big] - \frac{3}{2}\,m_t^2\,B_0\left(-s;m_t,m_t\right). \qquad (11)
\end{aligned}
$$

2.1   Running couplings

We now consider three parameters, the e.m. coupling constant e, the $SU(2)$ coupling constant g and the sine of the weak mixing angle s_θ. At the tree level they are not independent, but rather they satisfy the relation $g^2 s_\theta^2 = e^2$. The running of the e.m. coupling constant is easily derived and gives

$$\frac{1}{e^2(s)} = \frac{1}{g^2 s_\theta^2} - \frac{1}{16\,\pi^2}\,\Pi_{\gamma\gamma}(s). \qquad (12)$$

However, we have a natural scale to use since at $s = 0$ we have the fine structure constant at our disposal. Therefore, the running of $e^2(s)$ is completely specified in terms of α by

$$\frac{1}{e^2(s)} = \frac{1}{4\,\pi\alpha}\left[1 - \frac{\alpha}{4\,\pi}\,\Pi(s)\right], \qquad \text{with}\quad \Pi(s) = \Pi_{\gamma\gamma}(s) - \Pi_{\gamma\gamma}(0). \qquad (13)$$

For the running of g^2 we derive a similar equation:

$$\frac{1}{g^2(s)} = \frac{1}{g^2} - \frac{1}{16\,\pi^2}\,\Pi_{3Q}(s). \qquad (14)$$

The running of the third parameter, $s_\theta^2(s)$, is now fixed by

$$s_\theta^2(s) = \frac{e^2(s)}{g^2(s)}. \qquad (15)$$

2.2   Propagator functions.

The re-summed propagators for the vector bosons are:

$$G_\gamma(p^2) = \left\{p^2 - S_{\gamma\gamma}(p^2) - \frac{\left[S_{Z\gamma}(p^2)\right]^2}{p^2 + M_0^2 - S_{ZZ}(p^2)}\right\}^{-1},$$

$$G_{z\gamma}(p^2) = \frac{S_{z\gamma}(p^2)}{\left[p^2 - S_{\gamma\gamma}(p^2)\right]\left[p^2 + M_0^2 - S_{zz}(p^2)\right] - \left[S_{z\gamma}(p^2)\right]^2},$$

$$G_z(p^2) = \left\{ p^2 + M_0^2 - S_{zz} - \frac{\left[S_{z\gamma}(p^2)\right]^2}{p^2 - S_{\gamma\gamma}(p^2)} \right\}^{-1},$$

$$G_w(p^2) = \left[p^2 + M^2 - S_{ww}(p^2) \right]^{-1}. \tag{16}$$

The quantity $M_0 = M/c_\theta$ is the bare Z mass. An essential ingredient of the construction is represented by the location of the complex poles; they are determined by the following two equations:

$$p_w = M^2 - S_{ww}(p_w),$$

$$p_z = M_0^2 - Z(p_z), \qquad \text{where} \quad Z(s) = S_{zz}(s) - \frac{\left[S_{z\gamma}(s)\right]^2}{s + S_{\gamma\gamma}(s)}. \tag{17}$$

Substituting Eq.(17) into the expressions for the propagators, Eq.(16), we see that all ultraviolet divergences not proportional to p^2 cancel. We obtain

$$G_z(s) = \left[-s + p_z - Z(s) + Z(p_z) \right]^{-1},$$

$$G_w(s) = \left[-s + p_w - S_{ww}(s) + S_{ww}(p_w) \right]^{-1},$$

$$G_{z\gamma}(s) = -\frac{S_{z\gamma}(s)}{s + S_{\gamma\gamma}(s)} G_z(s),$$

$$G_\gamma(s) = -\frac{1}{s + S_{\gamma\gamma}(s)} + \left[\frac{S_{z\gamma}(s)}{s + S_{\gamma\gamma}(s)} \right]^2 G_z(s). \tag{18}$$

Using the self-energies and the running parameters we can write

$$1 + \frac{Z(s)}{s} = \frac{g^2}{c_\theta^2}\left[\frac{c^2(s)}{g^2(s)} + \frac{1}{16\,\pi^2}\frac{f_z(s)}{s} \right],$$

$$1 + \frac{S_{ww}(s)}{s} = g^2\left[\frac{1}{g^2(s)} + \frac{1}{16\,\pi^2}\frac{f_w(s)}{s} \right], \tag{19}$$

As a result, the vector boson propagators are now expressed as

$$G_w(s) = -\frac{g^2(s)}{g^2}\frac{\omega_w(s)}{s}, \qquad G_z(s) = -\frac{c_\theta^2}{g^2}\frac{g^2(s)}{c^2(s)}\frac{\omega_z(s)}{s},$$

$$G_{z\gamma}(s) = \frac{s_\theta}{c_\theta}\left[1 - \frac{s^2(s)}{s_\theta^2} \right] G_z(s),$$

$$G_\gamma(s) = \frac{e^2(s)}{e^2} + \frac{s_\theta^2}{c_\theta^2}\left[1 - \frac{s^2(s)}{s_\theta^2} \right]^2 G_z(s), \tag{20}$$

where the propagation functions are

$$\omega_w^{-1}(s) = 1 - \frac{g^2(s)}{s}\left\{ \frac{p_w}{g^2(p_w)} - \frac{1}{16\,\pi^2}\left[f_w(s) - f_w(p_w) \right] \right\},$$

$$\omega_z^{-1}(s) \;=\; 1 - \frac{g^2(s)}{c^2(s)s}\left\{\frac{c^2(p_z)}{g^2(p_z)}\,p_z - \frac{1}{16\,\pi^2}\Big[f_z(s)-f_z(p_w)\Big]\right\}. \quad (21)$$

The ω-functions are ultraviolet finite since the ultraviolet poles in f_z and f_w do not depend on the scale. The result of including vector boson transitions is illustrated schematically by the following diagrams (Eq.(22)):
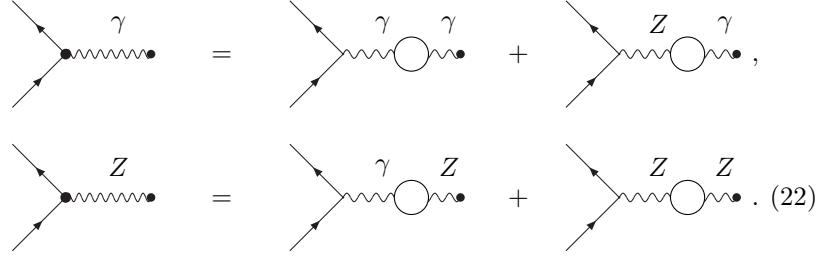

Here the open circles denote re-summed propagators and the dot a vertex. The dot on the right-hand side of the diagrams indicates that the corresponding leg is not amputated, i.e. that the propagator is included.

2.3 A simple recipe for implementing the Fermion-Loop scheme.

There is a simple recipe for implementing the Fermion-Loop scheme: take any process where the external sources are physical and on-shell, eg fermionic currents, then the complete procedure for the re-summation of self-energies and transitions, for the inclusion of running couplings and of re-summed propagators amounts to rewrite the corresponding Born amplitude in terms of re-summed, running, quantities without the inclusion of transitions like the $\gamma - Z$ one. Let us consider one example in more detail. First, we split the 20 Feynman diagrams of the CC20 family into the nine diagrams of Fig. 1, characterized by the presence of a t-channel photon or Z-boson, and the rest

$$\mathrm{CC20} = \mathrm{CC20}_{\gamma\oplus Z} + \mathrm{CC20_R}. \quad (23)$$
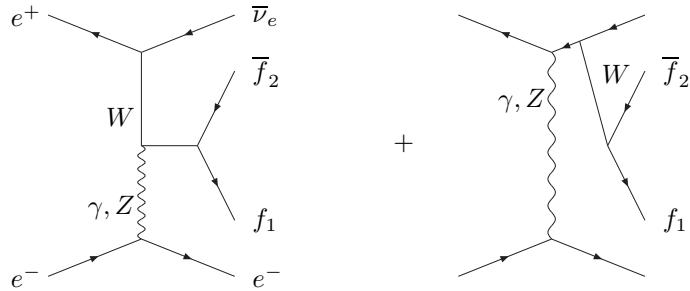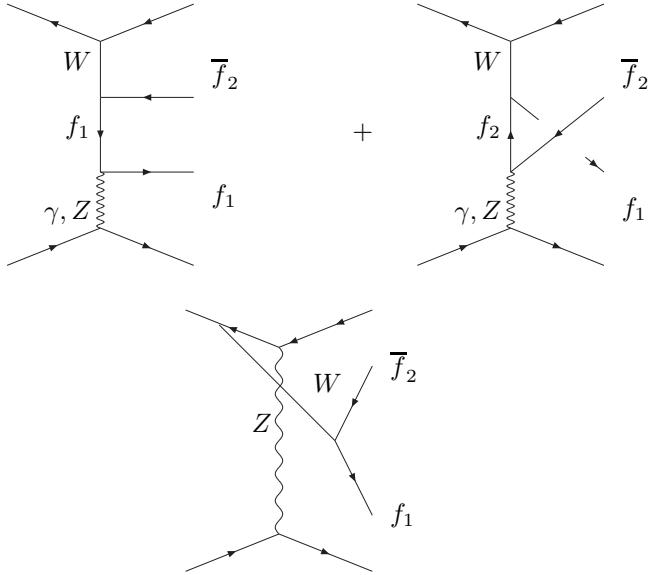
Figure 1: The CC20$_{\gamma \oplus Z}$ family of diagrams.

Momenta are assigned as follows:

$$e^+(p_+)\,e^-(p_-) \to e^-(q_-)\,\overline{\nu}_e(q_+)\,u(k)\,\overline{d}(\overline{k}), \quad Q_\pm = p_\pm - q_\pm. \qquad (24)$$

First we consider the diagram with the non-abelian coupling and generalize it to have re-summed W-boson propagators and $\gamma - \gamma, Z - \gamma$ (1a) transitions and $Z - Z, \gamma - Z$ (1b) transitions. The corresponding amplitudes will then become

$$M_1 = \mathrm{M}_{1\mathrm{a}\mu} \otimes (-igs_\theta)\,\gamma^\mu + \mathrm{M}_{1\mathrm{b}\mu} \otimes \frac{ig}{2\,c_\theta}\gamma^\mu \left(2\,s_\theta^2 - \frac{1}{2}\gamma_+ \right), \qquad (25)$$

where $\gamma_\pm = 1 \pm \gamma_5$ and the sub-amplitudes are

$$\begin{aligned}
\mathrm{M}_{1\mathrm{a}\mu} &= \frac{1}{8}\frac{G_s G_t}{g^3 s_\theta} \frac{\omega_W^s \omega_W^t}{(p_- - q_+)^2 (k+\overline{k})^2 Q_-^2} \left[e^2(T) - \frac{G_T}{c^2(T)}\,\omega_Z^T \Delta_\gamma \right] \\
&\times \quad \gamma^\alpha \gamma_+ \otimes \gamma^\beta \gamma_+ V_{\mu\alpha\beta}^0,
\end{aligned}$$

$$\begin{aligned}
\mathrm{M}_{1\mathrm{b}\mu} &= -\frac{1}{8}\frac{G_s G_t}{g^3} \frac{\omega_W^s \omega_W^t}{(p_- - q_+)^2 (k+\overline{k})^2 Q_-^2}\,G_T \omega_Z^T \\
&\times \quad \gamma^\alpha \gamma_+ \otimes \gamma^\beta \gamma_+ V_{\mu\alpha\beta}^0, \qquad\qquad\qquad\qquad (26)
\end{aligned}$$

and where $V_{\mu\alpha\beta}^0$ is the tree-level non-abelian coupling. Furthermore,

$$\Delta_\gamma = s_\theta^4 - 2\,s^2(T)s_\theta^2 + s^4(T) + s_\theta^2 c_\theta^2 - s^2(T)c_\theta^2. \qquad (27)$$

10

In the previous equations we have introduced the following quantities:

$$
\begin{aligned}
T &= -(p_- - q_-)^2, \\
G_T &= g^2(T), \quad G_t = g^2((p_+ - q_+)^2), \quad G_s = g^2((k + \overline{k})^2). \quad (28)
\end{aligned}
$$

Our sign convention is such that

$$
g^2(s) \equiv g^2(p^2)\,|_{p^2=-s}\,. \quad (29)
$$

In the same way we have introduced ω_W^s, the propagation function for a W-boson with $p^2 = -(k + \overline{k})^2$. When we combine the two results we obtain

$$
\begin{aligned}
M &= -\frac{i}{8}\frac{G_s G_t}{g^2}\frac{\omega_W^s \omega_W^t}{(p_- - q_+)^2(k + \overline{k})^2 Q_-^2}\left\{e^2(T)\gamma^\mu\right. \\
&+ \left.\frac{1}{4}G_T \omega_Z^T \gamma^\mu \left[4\,s^2(T) - \gamma_+\right]\right\} \otimes \gamma^\alpha \gamma_+ \otimes \gamma^\beta \gamma_+ \, V_{\mu\alpha\beta}^0. \quad (30)
\end{aligned}
$$

The only remaining bare quantity is an overall $1/g^2$ factor which, as discussed in Sect. 7, is essential in performing the renormalization of the amplitude. The rest is exactly the sum of two Born-like diagrams with γ and Z exchange and with running parameters instead of bare ones. In the following we will denote by Q_f the charge of the fermion. Similarly, for the other diagrams we obtain

$$
\begin{aligned}
M_2 &= M_{2a\mu} \otimes (-igs_\theta)\,\gamma^\mu + M_{2b\mu} \otimes \frac{ig}{2\,c_\theta}\gamma^\mu\left(2\,s_\theta^2 - \frac{1}{2}\gamma_+\right) \\
&= \frac{i}{8}\frac{G_s e^2(T)}{(p_- + Q_-)^2(k + \overline{k})^2 Q_-^2}\,\omega_W^s\,\gamma^\mu\,(\slashed{p}_- + \slashed{Q}_-) \otimes \gamma^\alpha \gamma_+ \otimes \gamma^\alpha \gamma_+ \otimes \gamma^\mu \\
&- \frac{i}{8}\frac{G_s G_T}{(p_- + Q_-)^2(k + \overline{k})^2 Q_-^2}\,\omega_W^s \omega_Z^T\,\frac{G_t}{16\,c^2(T)} \\
&\times \quad \gamma^\mu\left(4\,s^2(T) - \gamma_+\right)(\slashed{p}_- + \slashed{Q}_-)\,\gamma^\alpha \gamma_+ \otimes \gamma^\alpha \gamma_+ \otimes \gamma^\mu\left(4\,s^2(T) - \gamma_+\right),
\end{aligned}
$$

$$
\begin{aligned}
M_3 &= M_{3a\mu} \otimes (-igs_\theta)\,\gamma^\mu + M_{3b\mu} \otimes \frac{ig}{2\,c_\theta}\gamma^\mu\left(2\,s_\theta^2 - \frac{1}{2}\gamma_+\right) \\
&= \frac{i}{8}\frac{G_t e^2(T)}{Q_+^2 Q_-^2}\,Q_u\,\omega_W^t\,\gamma^\alpha \gamma_+ \otimes \gamma^\mu\frac{\slashed{Q}_- - \slashed{k} - i\,m_u}{(Q_- - k)^2 + m_u^2}\,\gamma^\alpha \gamma_+ \otimes \gamma^\mu \\
&+ \frac{i}{8}\frac{G_t G_T}{16\,c^2(T)\,Q_+^2 Q_-^2}\,\omega_W^t \omega_Z^T\,\gamma^\alpha \gamma_+ \\
&\otimes \quad \left(\gamma_+ - 4\,Q_u s^2(T)\right)\frac{\slashed{Q}_- - \slashed{k} - i\,m_u}{(Q_- - k)^2 + m_u^2}\,\gamma^\alpha \gamma_+ \otimes \gamma^\mu\left(4\,s^2(T) - \gamma_+\right),
\end{aligned}
$$

$$
\begin{aligned}
M_4 &= M_{4a\mu} \otimes (-igs_\theta)\,\gamma^\mu + M_{4b\mu} \otimes \frac{ig}{2\,c_\theta}\gamma^\mu\left(2\,s_\theta^2 - \frac{1}{2}\gamma_+\right) \\
&= \frac{i}{8}\frac{G_t e^2(T)}{Q_+^2 Q_-^2}\,Q_d\,\omega_W^t\,\gamma^\alpha \gamma_+ \otimes \gamma^\mu\frac{\slashed{Q}_- - \overline{\slashed{k}} + i\,m_d}{(Q_- - \overline{k})^2 + m_d^2}\,\gamma^\alpha \gamma_+ \otimes \gamma^\mu
\end{aligned}
$$

11

$$+ \quad \frac{i}{8} \frac{G_t G_T}{16 \, c^2(T) \, Q_+^2 Q_-^2} \, \omega_W^t \omega_Z^T \, \gamma^\alpha \gamma_+$$

$$\otimes \quad \left(-\gamma_+ - 4 \, Q_d s^2(T) \right) \frac{\slashed{Q}_- - \overline{\slashed{k}} + i \, m_d}{(Q_- - \overline{k})^2 + m_d^2} \, \gamma^\alpha \gamma_+ \, \otimes \gamma^\mu \, \left(4 \, s^2(T) - \gamma_+ \right),$$

$$M_5 \quad = \quad \mathrm{M}_{5\mathrm{a}\mu} \otimes (-igs_\theta) \, \gamma^\mu + \mathrm{M}_{5\mathrm{b}\mu} \otimes \frac{ig}{2 \, c_\theta} \gamma^\mu \left(2 \, s_\theta^2 - \frac{1}{2} \gamma_+ \right)$$

$$= \quad -\frac{i}{8} \frac{G_s G_t}{16 \, c^2(T) \, (k + \overline{k})^2 (Q_- - q_+)^2 Q_-^2} \, \omega_W^s \omega_Z^T$$

$$\times \quad \gamma^\alpha \gamma_+ \, (\slashed{Q}_- - \slashed{q}_+) \, \gamma^\mu \gamma_+ \otimes \gamma^\mu \, \left(4 \, s^2(T) - \gamma_+ \right) \otimes \gamma^\alpha \gamma_+. \tag{31}$$

This set of equations proves the assertion that Fermion-Loop is exactly an improved, gauge preserving, Born approximation.

To complete the construction of the Fermion-Loop scheme one must include the one-loop fermionic vertices. At the Born level the $\gamma W^+ W^-$ and the $Z W^+ W^-$ vertices are the same, once we have factorized s_θ and c_θ in front of them. For one-loop corrected vertices this is no longer true and one may wonder whether this fact spoils the transition from bare quantities to re-summed ones.

The lowest order interaction for $V(P) \to W^+(q_+) W^-(q_-)$ is specified by the tensor

$$V_{\mu\alpha\beta}^0 \, (P; q_+, q_-) \quad = \quad \delta_{\mu\beta} \, (P - q_-)_\alpha + \delta_{\alpha\beta} \, (q_- - q_+)_\mu + \delta_{\mu\alpha} \, (q_+ - P)_\beta \tag{32}$$

At the one-loop level we need seven independent form-factors, if the external sources are physical; they are as follows:

$$V_{\mu\alpha\beta}^1 \quad = \quad \frac{g^2 s_\theta}{16 \, \pi^2} \sqrt{s} \sum_{i=1,7} I_i \, W_{\mu\alpha\beta}^i,$$

$$W_{\mu\alpha\beta}^1 \quad = \quad \frac{4}{\sqrt{s}} \left[\delta_{\alpha\beta} q_{-\mu} + \delta_{\mu\beta} P_\alpha + \delta_{\mu\alpha} q_{+\beta} \right],$$

$$W_{\mu\alpha\beta}^{2,3} \quad = \quad \frac{2}{\sqrt{s}} \left[\delta_{\mu\beta} P_\alpha \pm \delta_{\mu\alpha} q_{+\beta} \right],$$

$$W_{\mu\alpha\beta}^4 \quad = \quad \frac{2}{s\sqrt{s}} \, q_{-\mu} P_\alpha q_{+\beta},$$

$$W_{\mu\alpha\beta}^{5,6} \quad = \quad \frac{1}{\sqrt{s}} \left[\varepsilon \, (q_+, \mu, \alpha, \beta) \pm \varepsilon \, (q_-, \mu, \alpha, \beta) \right],$$

$$W_{\mu\alpha\beta}^7 \quad = \quad \frac{1}{s\sqrt{s}} \varepsilon \, (q_-, q_+, \alpha, \beta) \, q_{+\mu}, \tag{33}$$

where we have introduced a special notation,

$$\varepsilon(a, b, c, d) = \varepsilon^{\mu\nu\alpha\beta} \, a_\mu b_\nu c_\alpha d_\beta. \tag{34}$$

For a massless fermion generation there is no difference between the Z and γ coefficients once we have factorized $c_\theta(s_\theta)$ in front of the full vertex. For the

third generation we have the same for all loops except the $(m_t, 0, m_t)$ one, where the difference between Z and γ is given by the following relations:

$$I_i^{ZWW} = I_i^{\gamma WW} + \frac{1}{c_\theta^2}\, \Delta I_i, \qquad i = 1, \dots, 7 \tag{35}$$

where the explicit expression for the extra term, ΔI_i, is of no concern here. The amplitude M_1, containing the non-abelian coupling, retains its structure when we substitute $V_{\gamma WW}^0$ with $V_{\gamma WW}^1$ and we have additional contributions

$$
\begin{aligned}
\delta_i &= \frac{g^3 \Delta I_i}{c_\theta^2}\, c_\theta\, G_z \left[\frac{s_\theta}{c_\theta} \left(1 - \frac{s^2(T)}{s_\theta^2} \right) (-igs_\theta) \otimes \gamma^\mu \right. \\
&\quad + \left. \frac{ig}{2\, c_\theta} \otimes \gamma^\mu \left(2\, s_\theta^2 - \frac{1}{2}\, \gamma_+ \right) \right] \\
&= +\frac{i}{4} g^2\, \Delta_i\, \frac{G_T}{c^2(T)}\, \frac{\omega_z^T}{Q_-^2} \otimes \gamma^\mu \left(4\, s^2(T) - \gamma_+ \right).
\end{aligned}
\tag{36}
$$

Therefore, also for one-loop vertices we can write a Born-like amplitude and promote all bare quantities to running ones, i.e. we take the full one-loop corrected vertex of Eq.(33) with factors $s_\theta(c_\theta)$ factorized and also replace $1/c_\theta^2$ with $1/c^2(T)$ in the extra term of Eq.(35).

3 Re-summed propagators in the charged sector.

We now proceed to the construction of the Fermion-Loop scheme for non-conserved currents. The first step consists in re-deriving the propagators in a situation where all external fermion masses are kept. We work in the 't Hooft-Feynman gauge and compute the following transitions:

- The $W - W$ transition, $S_W^{\mu\nu}$ with

$$S_W^{\mu\nu} = \frac{g^2}{16\,\pi^2}\, \Sigma_W^{\mu\nu}, \quad \Sigma_W^{\mu\nu} = \Sigma_W^0\, \delta^{\mu\nu} + \Sigma_W^1\, p^\mu p^\nu. \tag{37}$$

We also introduce a special notation,

$$\Sigma_W^T = \Sigma_W^0 + p^2\, \Sigma_W^1. \tag{38}$$

- The $\phi - \phi$ transition, S_ϕ with

$$S_\phi = \frac{g^2}{16\,\pi^2}\, \Sigma_\phi. \tag{39}$$

- The $W - \phi$ and $\phi - W$ transitions,

$$S_{W\phi}^\mu = +\frac{g^2}{16\,\pi^2}\, \Sigma_{W\phi}\, ip^\mu, \quad S_{\phi W}^\mu = -\frac{g^2}{16\,\pi^2}\, \Sigma_{W\phi}\, ip^\mu. \tag{40}$$

13

Let us introduce indices $a, b, \ldots = 1, \ldots, 5$; the re-summation amounts to write the following equation:

$$
\begin{aligned}
\bar{\Delta}_{ab} &= \Delta_{ab} + \delta_{ac} S_{cd} \Delta_{db} + \ldots \\
&= \delta_{ac} \left(\delta_{cb} + S_{cd} \Delta_{db} + \ldots \right) = \Delta_{ac} X_{cb}, \\
X_{ab} &= (1 - S \Delta)^{-1}_{ab}.
\end{aligned}
\tag{41}
$$

Here we have made use of Born propagators given, in the 't Hooft-Feynman gauge, by

$$
\Delta_{WW}^{\mu\nu} = \frac{\delta_{\mu\nu}}{p^2 + M^2}, \qquad \Delta_{\phi\phi} = \frac{1}{p^2 + M^2}.
\tag{42}
$$

Examples of Dyson re-summation are as follows:

$$
\begin{aligned}
\bar{\Delta}_{WW}^{\mu\nu} &= \Delta_{WW}^{\mu\nu} + \Delta_{WW}^{\mu\alpha} S_W^{\alpha\beta} \delta_{WW}^{\beta\nu} + \ldots, \\
\bar{\Delta}_{\phi\phi} &= \Delta_{\phi\phi} + \Delta_{\phi\phi} S_\phi \Delta_{\phi\phi} + \ldots, \\
\bar{\Delta}_{W\phi}^{\mu} &= \Delta_{WW}^{\mu\alpha} S_{W\phi}^{\alpha} \Delta_{\phi\phi} + \ldots
\end{aligned}
\tag{43}
$$

. After performing the inversion of the matrix in Eq.(41), we obtain

$$
\bar{\Delta}_{WW}^{\mu\nu} = \frac{1}{p^2 + M^2 - S_W^0} \left[\delta^{\mu\nu} + \frac{S_W^1 + (S_{W\phi})^2 / (p^2 + M^2 - S_\phi)}{p^2 + M^2 - S_W^T - p^2 (S_{W\phi})^2 / (p^2 + M^2 - S_\phi)} p^\mu p^\nu \right]
\tag{44}
$$

The previous result can be cast into a simpler form when we use some important relation originating from Ward identities applied to two-point functions. The Ward identities for transitions in the charged sector are shown in Fig. 2 where we used the symbol =, attached to a vector boson line, to indicate multiplication by $i\, p_\mu$.
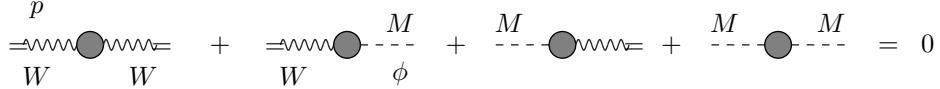
Figure 2: Example of Ward identities for transitions in the charged sector.

From the explicit expressions of these Ward identities we derive the following results:

$$
\begin{aligned}
S_{W\phi} &= \frac{M}{p^2} S_\phi, \qquad S_W^T = \frac{M^2}{p^2} S_\phi, \\
S_W^1 &= \frac{1}{p^2} \left(\frac{M^2}{p^2} S_\phi - S_W^0 \right).
\end{aligned}
\tag{45}
$$

With their help the re-summed W propagator becomes

$$
\bar{\Delta}_{WW}^{\mu\nu} = \frac{1}{p^2 + M^2 - S_W^0} \left(\delta^{\mu\nu} + \Delta_L p^\mu p^\nu \right),
$$

$$\Delta_L = \frac{1}{p^2} \left[M^2 S_\phi - p^2 S_W^0 + \frac{M^2 (S_\phi)^2}{p^2 + M^2 - S_\phi} \right]$$
$$\times \left[p^2 (p^2 + M^2) - M^2 S_\phi - \frac{M^2 (S_\phi)^2}{p^2 + M^2 - S_\phi} \right]^{-1}. \tag{46}$$

The re-summed W propagator satisfies the following identity:

$$p_\mu \bar{\Delta}_{WW}^{\mu\nu} = \frac{p^\nu}{p^2 + M^2 - \frac{M^2}{p^2} S_\phi \left[1 + S_\phi / (p^2 + M^2 - S_\phi) \right]}. \tag{47}$$

Similarly, we obtain the $W - \phi$ re-summed transition,

$$\bar{\Delta}_{W\phi}^\mu = i \frac{M p^\mu}{p^2} \frac{S_\phi}{p^2 + M^2 - S_\phi} \frac{1}{p^2 + M^2 - \frac{M^2}{p^2} S_\phi \left[1 + S_\phi / (p^2 + M^2 - S_\phi) \right]}, \tag{48}$$

and the $\phi - \phi$ re-summed transition,

$$\bar{\Delta}_{\phi\phi} = \frac{p^2 + M^2 - M^2/p^2 \, S_\phi}{(p^2 + M^2 - S_\phi)(p^2 + M^2 - M^2/p^2 \, S_\phi) - M^2/p^2 \, (S_\phi)^2}. \tag{49}$$

Before continuing we define some auxiliary quantities:

$$\bar{\Delta}_{WW}^{\mu\nu} = \frac{1}{p^2 + M^2 - S_W^0} \left(\delta^{\mu\nu} + \frac{N_v}{p^4 \, D} p^\mu p^\nu \right),$$
$$\bar{\Delta}_{W\phi}^\mu = i \frac{M p^\mu}{p^2} \frac{S_\phi}{(p^2 + M^2 - S_\phi) \, D},$$
$$\bar{\Delta}_{\phi\phi} = \frac{1}{p^2 + M^2 - S_\phi} \frac{N_s}{D}, \tag{50}$$

where we have introduced

$$D = p^2 + M^2 - \frac{M^2}{p^2} S_\phi \left[1 + \frac{S_\phi}{p^2 + M^2 - S_\phi} \right],$$
$$N_v = M^2 S_\phi - p^2 S_W^0 + \frac{M^2 (S_\phi)^2}{p^2 + M^2 - S_\phi},$$
$$N_s = p^2 + M^2 - \frac{M^2}{p^2} S_\phi. \tag{51}$$

We have seen that, for zero external masses, the re-summation of transitions has a very simple effect on the Born amplitude, it promotes all quantities to running ones and all bare couplings disappear from the amplitude itself. Is it possible to have a generalization of this phenomenon that accounts for non zero external masses? The answer to this question will be the subject of the next section.

4   The running W-boson mass.

Running couplings appear naturally in the one-loop corrected amplitude when we consider an $\mathcal{S}$-element, i.e. a transition between physical sources. Therefore, we start with some simple example to illustrate the generalization of the Fermion-Loop scheme. Consider the process $\nu_\mu \mu \to \overline{\nu}_e e$, as given in Fig. 3.
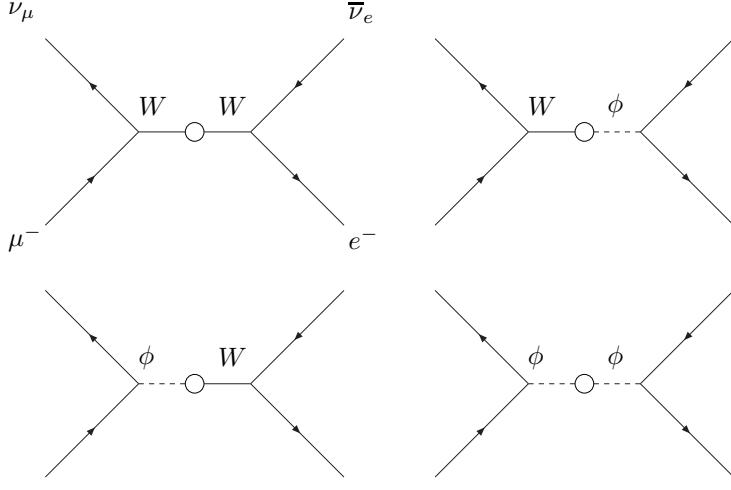

Figure 3: The process $\mu\nu_\mu \to e\overline{\nu}_e$.

For our purposes only four diagrams are relevant. They correspond to the inclusion of re-summed $W - W, W - \phi, \phi - W$ and $\phi - \phi$ transitions. We obtain

$$
\begin{aligned}
M &= \left(\frac{ig}{2\sqrt{2}}\right)^2 \mathcal{M}, \\
\mathcal{M} &= \gamma^\mu \gamma_+ \otimes \gamma^\nu \gamma_+ \, \bar{\Delta}^{\mu\nu}_{WW} + \frac{m_e}{M} \gamma^\mu \gamma_+ \otimes \gamma_+ \, \bar{\Delta}^\mu_{W\phi} \\
&\quad - \frac{m_\mu}{M} \gamma_- \otimes \gamma^\mu \gamma_+ \, \bar{\Delta}^\mu_{\phi W} - \frac{m_e m_\mu}{M^2} \gamma_- \otimes \gamma_+ \, \bar{\Delta}_{\phi\phi}.
\end{aligned}
\tag{52}
$$

The amplitude can we written as the sum of two terms, a familiar one where $m_e = m_\mu = 0$ and an *extra* contribution given by:

$$
\mathcal{M} = \frac{1}{p^2 + M^2 - S^0_W} \gamma^\mu \gamma_+ \otimes \gamma^\mu \gamma_+ + \mathcal{M}_{\text{extra}}.
\tag{53}
$$

16

After some lengthy but straightforward algebra, making use of the Dirac equation and of the relation,

$$D \left(p^2 + M^2 - S_\phi \right) = \left(p^2 + M^2 \right)^2 \left(1 - \frac{S_\phi}{p^2} \right), \tag{54}$$

the second term in Eq.(53) can be written as follows:

$$\mathcal{M}_{\text{extra}} = m_e m_\mu \, \gamma_- \otimes \gamma_+ \, \frac{1}{p^2 + M^2 - S_W^0} \, \frac{S_W^0 - p^2 - \frac{M^2}{p^2} \, S_\phi}{M^2 \left(p^2 - S_\phi \right)}. \tag{55}$$

There are two ingredients that we need to continue our construction of the Fermion-Loop scheme. First the complex W-pole. We start with the relation

$$- g^2 \Delta_W = -g^2 \left[p^2 + M^2 - S_W^0 (p^2) \right]^{-1} = \left[\frac{s - M^2}{g^2} + \frac{1}{16 \, \pi^2} \, \Sigma_W^0 \right]^{-1}, \tag{56}$$

and define the complex pole as a solution of

$$\frac{p_W - M^2}{g^2} + \frac{1}{16 \, \pi^2} \, \Sigma_W^0 \, (p_W) = 0. \tag{57}$$

Therefore, for the W propagator, we obtain

$$- g^2 \, \Delta_W = \left\{ \frac{s - p_W}{g^2(s)} + \frac{p_W}{16 \, \pi^2} \left[\Pi_{3Q} \, (p_W) - \Pi_{3Q} \, (s) \right] + \frac{1}{16 \, \pi^2} \left[f_W (s) - f_W (p_W) \right] \right\}^{-1}. \tag{58}$$

As expected, the position of the complex W-boson pole is solely fixed by the S_W^0-component of its self-energy. Next, we need a second ingredient, the running W-boson mass.

Definition 1 *The W-boson running mass is defined by the following equation:*

$$\frac{1}{M^2(p^2)} = \frac{1}{M^2} \, \frac{p^2 - S_W^0 + \frac{M^2}{p^2} \, S_\phi}{p^2 - S_\phi}. \tag{59}$$

It is not an independent quantity but, instead, it is related to the complex pole. To give a simple illustration of this relation, let us consider the case of massless fermions in the one-loop transitions. This means, in particular, $S_\phi = 0$. Therefore, we have

$$\frac{1}{M^2(p^2)} = \left(1 - \frac{S_W^0}{p^2} \right) \frac{1}{M^2}. \tag{60}$$

Using the fact that the bare mass and the complex pole are related by

$$M^2 = p_W + S_W^0 (p_W), \tag{61}$$

and combining it with the following two relations,

$$S_W^0 (p^2) = p^2 \, \frac{g^2}{16 \, \pi^2} \, \Pi_{3Q} \, (p^2), \qquad \frac{1}{g^2(p^2)} = \frac{1}{g^2} - \frac{1}{16 \, \pi^2} \, \Pi_{3Q} \, (p^2), \tag{62}$$

17

for a massless internal world we obtain

$$M^2(p^2) = \frac{g^2(p^2)}{g^2(p_W)}\, p_W, \qquad M^2(p_W) = p_W. \tag{63}$$

Equipped with this result, we can write $\mathcal{M}_{\text{extra}}$ as

$$\mathcal{M}_{\text{extra}} = \gamma^\mu \gamma_+ \otimes \gamma^\nu \gamma_+ \frac{1}{p^2 + M^2 - S_W^0} \frac{p^\mu p^\nu}{M^2(p^2)}. \tag{64}$$

To summarize our findings, the complete one-loop re-summation in the 't Hooft-Feynman gauge is equivalent to some *effective* unitary-gauge W-propagator. The whole amplitude can be written in terms of a W-boson exchange diagram, if we make use of the following effective propagator:

$$\Delta_{\text{eff}}^{\mu\nu} = \frac{1}{p^2 + M^2 - S_W^0} \left[\delta^{\mu\nu} + \frac{p^\mu p^\nu}{M^2(p^2)} \right]. \tag{65}$$

we obtain a similar, although a little more complicated, result also when the top quark mass is not neglected in loop corrections. We start with

$$S_W^0 = \frac{g^2}{16\,\pi^2} \left[p^2\, \Pi_{3Q}(p^2) + f_W(p^2) \right], \tag{66}$$

and derive

$$S_\phi = \frac{g^2}{16\,\pi^2} \frac{m_t^2}{M^2}\, f_\phi(p^2). \tag{67}$$

With $f_i(p^2) = p^2 \sigma_i(p^2)$, we end up with the following result for the running mass:

$$\frac{p_W}{M^2(p^2)} = \frac{g^2(p_W)}{g^2(p^2)} \left\{ 1 - \frac{g^2(p^2)}{16\,\pi^2} \left[\sigma_W(p^2) - \frac{m_t^2}{p^2}\, \sigma_\phi(p^2) \right] \right\}$$

$$\times\ \left\{ 1 - \frac{g^2(p_W)}{16\,\pi^2} \left[\sigma_W(p_W) + \frac{m_t^2}{p_W}\, \sigma_\phi(p^2) \right] \right\}^{-1}. \tag{68}$$

The above result is ultraviolet finite. Indeed we obtain

$$f_W(p^2)\,|_{\text{UV}} = -\frac{3}{2}\, m_t^2\, \frac{1}{\bar{\varepsilon}}, \tag{69}$$

where $\bar{\varepsilon}$ is the ultraviolet regulator,

$$\frac{1}{\bar{\varepsilon}} = \frac{2}{\varepsilon} - \gamma - \ln \pi, \tag{70}$$

and $\gamma = 0.577216$ is the Euler constant. From the explicit expression for S_ϕ, i.e.

$$S_\phi(p^2) = -\frac{3}{2} \left[A_0\,(m_t) + \left(p^2 + m_t^2 \right)\, B_0\left(p^2; m_t, 0 \right) \right], \tag{71}$$

we get the ultraviolet part of f_ϕ,

$$f_\phi(p^2)\,|_{\text{UV}} = -\frac{3}{2}\, p^2\, \frac{1}{\bar{\varepsilon}}, \tag{72}$$

18

giving a cancellation of the ultraviolet divergent term, $1/\bar{\varepsilon}$, inside Eq.(68).

There are several examples where one can show that external fermion masses can be easily included in the Fermion-Loop scheme. We simply promote all quantities to be running ones and use the unitary-gauge expression for the W-boson propagator, but with a running mass. Some of these examples are very instructive, since they clearly show how the strategy works only for $\mathcal{S}$-matrix elements, i.e. for amputated Green's function with on-shell and properly renormalized external sources.

Consider $W^+(q) + \gamma(k) \rightarrow u\overline{d}$, a component of the single-W process. We have an amplitude that can be written as follows:

$$M_{\mu\alpha} = V^0_{\mu\alpha\beta} \, \frac{\gamma^\beta \gamma_+}{p^2 + M^2 - S^0_W} + i \, \mathcal{M}_{\mu\alpha} \, (m_u \gamma_+ - m_d \gamma_-) \,. \tag{73}$$

The first result that we obtain is for a situation where the incoming particles are physical, i.e.

$$k^2 = 0, \quad q^2 = -M^2, \qquad k \cdot e(k) = 0, \quad q \cdot \varepsilon(q) = 0, \tag{74}$$

where $e_\mu(k)$ and $\varepsilon_\alpha(q)$ are the photon and the W polarization vectors. In this case we find

$$\mathcal{M}_{\mu\alpha} = \frac{V^0_{\mu\alpha\beta}}{p^2 + M^2 - S^0_W} \, \frac{p^\beta}{M^2(p^2)}, \tag{75}$$

where $p = q + k$. To go further, we only require conservation of the e.m. current, $k \cdot e(k) = 0$, but not the mass-shell condition $k^2 = 0$. In this case, even for $m_e = 0$, a residual term remains:

$$\delta_{\mu\alpha} \, p^2 \, \frac{M^2 + q^2 - k^2}{M^2 \, (p^2 + M^2) \, (p^2 - S_\phi)} \,. \tag{76}$$

To understand the mechanism of cancellation we embed the sub-process (four diagrams) corresponding to the annihilation $W^+\gamma \rightarrow u\overline{d}$ into $e^+\gamma \rightarrow \overline{\nu}_e u\overline{d}$. We obtain the following result:


annihilation   **bremsstrahlung**
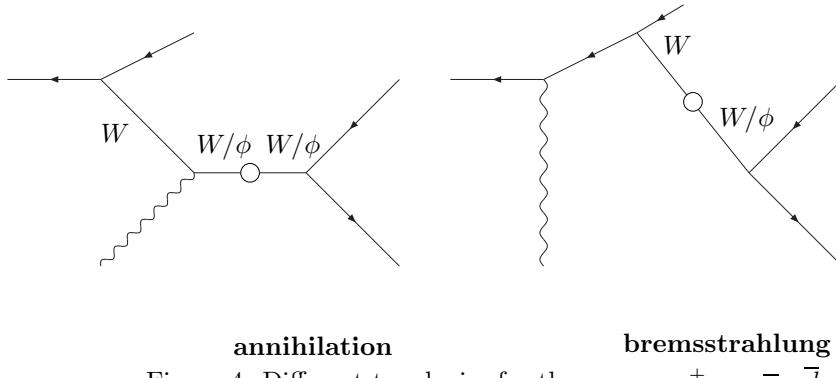
Figure 4: Different topologies for the process $e^+\gamma \rightarrow \overline{\nu}_e u\overline{d}$.

$$M_\mu^{\mathrm{ann}} = \frac{V_{\mu\alpha\beta}^0}{p^2 + M^2 - S_W^0} \left[\delta^{\beta\lambda} + \frac{p^\beta p^\lambda}{M^2(p^2)} \gamma^\alpha\gamma_+ \otimes \gamma^\lambda\gamma_+ \right.$$

$$+ \; \delta_{\mu\alpha} \frac{p^2}{M^2} \frac{M^2 + q^2 - k^2}{(p^2 + M^2)(p^2 - S_\phi)} \gamma^\alpha\gamma_+ \otimes \not{p}\gamma_+. \qquad (77)$$

Some unwanted term remains. However, there is another contribution, where the $u\overline{d}$ pair is emitted by a W-boson that, in turn, is coming from the splitting $e \to \nu_e W$, see Fig. 4. Here, in the limit $m_e = 0$, two diagrams contribute, a $W - W$ propagator and a $W - \phi$ propagator. We find

$$M_\mu^{\mathrm{brem}} = \frac{1}{p^2 + M^2 - S_W^0} \left[\delta_{\alpha\beta} + \frac{p_\alpha p_\beta}{M^2(p^2)} \right] \gamma^\mu \frac{i(\not{p}_+ + \not{k})}{(p_+ + k)^2} \gamma^\alpha\gamma_+ \otimes \gamma^\beta\gamma_+$$

$$- \; \frac{p^2}{M^2} \frac{1}{(p^2 + M^2)(p^2 - S_\phi)} \gamma^\mu \frac{i(\not{p}_+ + \not{k})}{(p_+ + k)^2} (\not{p}_+ + \not{k}) \gamma_+ \otimes \not{p}\gamma_+. (78)$$

The two extra terms coming from Eqs.(77–78) add up to something proportional to k^2, therefore vanishing for $k^2 = 0$. This line of arguments is correct only as long as we neglect re-summation in the t-channel W-propagator.

5 Ward identities for single-W.

Before proving the relevant U(1) Ward identity for the single-W amplitude we recall that the total of 20 Feynman diagrams is, first of all, split into a 10 s-channel part and a 10 t-channel part. In the latter part, one diagram has a W-exchange, five a Z-exchange and four a γ-exchange. As we have shown, this picture is not changed by the inclusion of one-loop fermionic corrections: when all transitions are properly taken into account, we still end up with the $1 - 5 - 4$ subdivision described above, as long as the fermion–anti-fermion-vector–boson couplings are described in terms of the re-summed expressions.

This $s \oplus t$ splitting is a gauge invariant one, although we can further restrict the number of diagrams. The argument is as follows: Take $e^+\mu^- \to \overline{\nu}_e\mu^- u\overline{d}$. Only the CC20 t-channel diagrams contribute and, since these are all diagrams that we need, this set is gauge invariant. Next, take $e^+\nu_\mu \to \overline{\nu}_e\nu_\mu u\overline{d}$. Its is, again, only t-channel but the photon does not contribute, so that we have $10 - 4 = 6$ diagrams. Moreover, if one writes any Z current as $J = J_Q + J_L$ (with J_L proportional to γ_+), only J_L contributes here, because of the neutrinos, so that we have five J_L^Z diagrams plus one W diagram that form a gauge invariant set. Since the whole t-channel is gauge invariant, the eight diagrams, four with photons and four with J_Q^Z, must form a gauge invariant sub-set. This remains true for one-loop corrections when writing everything in terms of running objects and including vertices.

The gauge invariance property that we are referring to is the full SU(2) one. However, the first step is to prove the simpler U(1) gauge invariance or,

stated differently, we write the amplitude squared for the four photon-mediated t-channel diagrams as the product of a leptonic tensor $L_{\mu\nu}$ and of a single-W tensor, $W_{\mu\nu}$. Successively, we must be able to prove that $Q^{-\mu}W_{\mu\nu} = Q^{-\nu}W_{\mu\nu} = 0$ (where Q_- is the photon momentum), so that the only terms that survive in the product $L_{\mu\nu}W_{\mu\nu}$ are those proportional to Q^2_- or to m^2_e, giving rise to the familiar contributions to the cross-section. Either logarithmically enhanced, $1/Q^2_-$, or the sub-leading, constant one, m^2_e/Q^4_-.

To be more specific, we give the explicit expression for $L_{\mu\nu}$. With momenta assignment $e^+(p_+)e^-(p_-) \to e^-(q_-)\overline{\nu}_e(q_+)u(k)\overline{d}(\overline{k})$, we have

$$L_{\mu\nu} = 2\,Q^2_-\delta_{\mu\nu} + 4\,(p_{-\mu}q_{-\nu} + q_{-\mu}p_{-\nu}). \tag{79}$$

U(1) gauge invariance, yet to be proven, requires that the following decomposition holds for $W_{\mu\nu}$,

$$
\begin{aligned}
W_{\mu\nu} &= W_1\left[-\delta_{\mu\nu} + \frac{Q_{-\mu}Q_\nu}{Q^2_-}\right] - W_2\,\frac{Q^2_-}{(p_+\cdot Q_-)^2}\left(p_{+\mu} - \frac{p_+\cdot Q_-}{Q^2_-}Q_{-\mu}\right) \\
&\times \left(p_{+\nu} - \frac{p_+\cdot Q_-}{Q^2_-}Q_{-\nu}\right).
\end{aligned}
\tag{80}
$$

After contraction, and for $X \to 0$, we obtain

$$\frac{1}{4}\,L_{\mu\nu}W^{\mu\nu} = W_1\left[2\,\frac{m^2_e}{(Xys)^2} - \frac{1}{Xys}\right] + 2\,W_2\,\frac{y-1}{Xy^3s} + \mathcal{O}(X), \tag{81}$$

where we have introduced the variable y, equivalent to the fraction of the electron energy carried by the photon, and

$$
\begin{aligned}
p_+\cdot Q_- &= \frac{1}{2}\left[m^2_e - (X+1)\,ys\right], \\
Q^2_- &= X\,ys, \qquad (p_+ + p_-)^2 = -ys.
\end{aligned}
\tag{82}
$$

The decomposition of Eq.(80) requires that we can prove a U(1) Ward identity. This we will do without performing any approximation, within the framework of the Fermion-Loop scheme. One has to be fully aware that the seven tensor structures, introduced in Eq.(33), are not enough to describe the situation, simply because currents are not-conserved.

Before giving the most general structure for the vertex, we will introduce the invariants needed to describe the sub-process, $e^+(p_+)\gamma(Q_-) \to \overline{\nu}_e(q_+)u(k)\overline{d}(\overline{k})$. They are specified in the following list:

$$
\begin{aligned}
p_+\cdot Q_- &= \frac{1}{2}\left(m^2_e - Q^2_- - y\,s\right), \\
k\cdot\overline{k} &= \frac{1}{2}\left(-s' + m^2_u + m^2_d\right), \\
Q_-\cdot k &= \frac{1}{2}\left(Q^2_- - m^2_u + t\right),
\end{aligned}
$$

$$p_+ \cdot \overline{k} = \frac{1}{2}\left(-m_e^2 - m_d^2 + t'\right),$$

$$Q_- \cdot \overline{k} = \frac{1}{2}\left(Q_-^2 - m_d^2 + u\right),$$

$$p_+ \cdot k = \frac{1}{2}\left(-m_e^2 - m_u^2 + u'\right),$$

$$p_+ \cdot q_+ = \frac{1}{2}\left(\kappa_+ - m_e^2\right),$$

$$Q_- \cdot q_+ = \frac{1}{2}\left(Q_-^2 + \kappa_-\right),$$

$$\overline{k} \cdot q_+ = \frac{1}{2}\left(-m_d^2 + \zeta_+\right),$$

$$k \cdot q_+ = \frac{1}{2}\left(m_u^2 + \zeta_-\right), \tag{83}$$

The linearly independent invariants are:

$$t = \tau ys, \quad s' = x_2 ys, \quad \kappa_- = zys,$$

$$\zeta_- = (x_2 - x_1)\, ys, \quad Q_-^2 = Xys, \tag{84}$$

with the following solution for the remaining ones:

$$\kappa_+ = m_e^2 + y\, s(-1 - X + x_2 - z),$$

$$u = m_e^2 + m_u^2 + m_d^2 + y\, s(-1 - \tau - 2X - z),$$

$$u' = m_e^2 + 2\, m_u^2 + m_d^2 + y\, s(-\tau - X - x_1),$$

$$t' = -m_u^2 + y\, s(\tau + X + x_1 - x_2 + z),$$

$$\zeta_+ = -m_u^2 + m_d^2 + y\, s(-1 + x_1). \tag{85}$$

We introduce another auxiliary quantity

$$Q_+^2 = -\, Yys, \tag{86}$$

and also the vector $Q = Q_+ + Q_- = k + \overline{k}$. In the one-loop corrected $\gamma W^+ W^-$ vertex of Fig. 5 all vector-boson lines are off mass-shell and non-conserved.
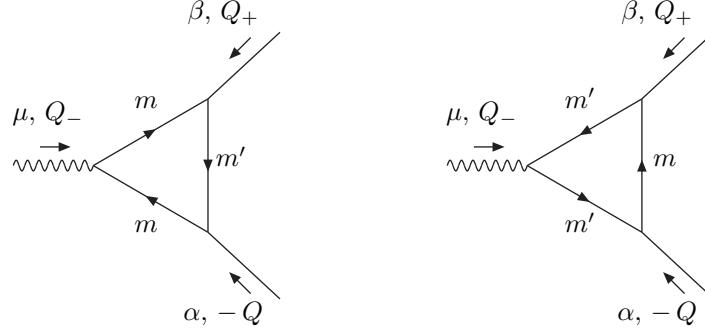

Figure 5: The $\gamma W^+ W^-$ vertex.

22

For the e.m. current this is a consequence of the fact that we are computing a Ward identity and not an amplitude. As a consequence, several more structures are needed and we list them below:

$$W^1_{\mu\alpha\beta} = 4\mathcal{N}\left[\delta_{\alpha\beta}Q_{+\mu} + \delta_{\mu\beta}Q_{-\alpha} - \delta_{\mu\alpha}Q_\beta\right],$$

$$W^{2,3}_{\mu\alpha\beta} = 2\mathcal{N}\left[\delta_{\mu\beta}Q_\alpha \mp \delta_{\mu\alpha}Q_\beta\right],$$

$$W^4_{\mu\alpha\beta} = 2\mathcal{N}\,\delta_{\alpha\beta}Q_{-\mu},$$

$$W^5_{\mu\alpha\beta} = 2\mathcal{N}\,\delta_{\mu\alpha}Q_{+\beta},$$

$$W^6_{\mu\alpha\beta} = 2\mathcal{N}\,\delta_{\mu\beta}Q_{+\alpha},$$

$$W^7_{\mu\alpha\beta} = -2\mathcal{N}\,\delta_{\mu\beta}Q_\alpha,$$

$$W^8_{\mu\alpha\beta} = -2\mathcal{N}\,\delta_{\alpha\beta}Q_\mu,$$

$$W^9_{\mu\alpha\beta} = 2\mathcal{N}^3 Q_{+\mu}Q_{-\alpha}Q_\beta,$$

$$W^{10}_{\mu\alpha\beta} = 2\mathcal{N}^3\,Q_{-\mu}Q_{-\alpha}Q_\beta,$$

$$W^{11}_{\mu\alpha\beta} = -2\mathcal{N}^3\,Q_{+\mu}Q_{+\alpha}Q_{+\beta},$$

$$W^{12}_{\mu\alpha\beta} = 2\mathcal{N}^3\,Q_{+\mu}Q_{+\alpha}Q_\beta,$$

$$W^{13}_{\mu\alpha\beta} = 2\mathcal{N}^3\,Q_{+\mu}Q_\alpha Q_{+\beta},$$

$$W^{14}_{\mu\alpha\beta} = -2\mathcal{N}^3\,Q_{+\mu}Q_\alpha Q_\beta,$$

$$W^{15}_{\mu\alpha\beta} = 2\mathcal{N}^3\,Q_\mu Q_{+\alpha}Q_{+\beta},$$

$$W^{16}_{\mu\alpha\beta} = -2\mathcal{N}^3\,Q_\mu Q_{+\alpha}Q_\beta,$$

$$W^{17}_{\mu\alpha\beta} = -2\mathcal{N}^3\,Q_\mu Q_\alpha Q_{+\beta},$$

$$W^{18}_{\mu\alpha\beta} = 2\mathcal{N}^3\,Q_\mu Q_\alpha Q_\beta,$$

$$W^{19}_{\mu\alpha\beta} = -2\mathcal{N}\,\varepsilon(Q_-,\mu,\alpha,\beta),$$

$$W^{20}_{\mu\alpha\beta} = -2\mathcal{N}\left[\varepsilon(Q_-,\mu,\alpha,\beta) + 2\,\varepsilon(Q_+,\mu,\alpha,\beta)\right],$$

$$W^{21}_{\mu\alpha\beta} = -\mathcal{N}^3\,\varepsilon(Q_+,Q_-,\alpha,\beta)Q_\mu,$$

$$W^{22}_{\mu\alpha\beta} = \mathcal{N}^3\,\varepsilon(Q_+,Q_-,\alpha,\beta)Q_{+\mu},$$

$$W^{23}_{\mu\alpha\beta} = \mathcal{N}^3\left[\varepsilon(Q_+,Q_-,\mu,\alpha)Q_{+\beta} - \varepsilon(Q_+,Q_-,\mu,\beta)Q_{+\alpha}\right],$$

$$W^{24}_{\mu\alpha\beta} = -\mathcal{N}^3\,\varepsilon(Q_+,Q_-,\mu,\beta)Q_\alpha. \tag{87}$$

The normalization factor is $\mathcal{N} = (Xys)^{-1/2}$. In presenting the vertices we face the usual problem of introducing a full *scalarization* of the result; that is, all results should be presented in terms of one-loop scalar-integral coefficient functions. This procedure is mandatory since the symmetry of the vertices has to be verified. However, there is no need to show the scalarized version of the

full vertices, only the corresponding contraction entering into the Ward identity is needed and, as a matter of fact, the latter is considerably simpler and much shorter. To summarize, we will not present explicitly the full vertex

$$V^1_{\mu\alpha\beta} = \frac{g^3 s_\theta}{16\,\pi^2}\,(Xys)^{1/2} \sum_{i=1,24} I_i\,W^i_{\mu\alpha\beta}, \tag{88}$$

but only the contraction. Actually, we cannot limit our analysis to the $\gamma W^+ W^-$ vertex but we must include also the $\gamma W^+ \phi^-, \gamma \phi^+ W^-$ and $\gamma \phi^+ \phi^-$ vertices. For $\gamma W^+ \phi^-$ the operators are as follows:

$$
\begin{aligned}
W^1_{\mu\beta} &= \delta_{\mu\beta}, \\
W^2_{\mu\beta} &= \mathcal{N}^2\,Q_{+\mu}Q_\beta, \\
W^3_{\mu\beta} &= -\mathcal{N}^2\,Q_{+\mu}Q_{+\beta}, \\
W^4_{\mu\beta} &= \mathcal{N}^2\,Q_\mu Q_{+\beta}, \\
W^5_{\mu\beta} &= -\mathcal{N}^2\,Q_\mu Q_\beta, \\
W^6_{\mu\beta} &= \mathcal{N}^2\,\varepsilon(Q_+, Q_-, \mu, \beta).
\end{aligned}
\tag{89}
$$

For $\gamma \phi^+ W^-$ the operators are as follows:

$$
\begin{aligned}
W^1_{\mu\alpha} &= \delta_{\mu\alpha}, \\
W^2_{\mu\alpha} &= \mathcal{N}^2\,Q_{+\mu}Q_\alpha, \\
W^3_{\mu\alpha} &= -\mathcal{N}^2\,Q_{+\mu}Q_{+\alpha}, \\
W^4_{\mu\alpha} &= \mathcal{N}^2\,Q_{+\mu}Q_\alpha, \\
W^5_{\mu\alpha} &= -\mathcal{N}^2\,Q_\mu Q_\alpha, \\
W^6_{\mu\alpha} &= \mathcal{N}^2\,\varepsilon(Q_+, Q_-, \mu, \alpha).
\end{aligned}
\tag{90}
$$

Finally for $\gamma \phi^+ \phi^-$ we have

$$W^1_\mu = Q_{+\mu}, \qquad W^2_\mu = Q_\mu. \tag{91}$$

For $\gamma W^+ W^-$ we will, therefore, present the contraction

$$
\begin{aligned}
V^{WW}_{\alpha\beta} &= Q^\mu_- V^1_{\mu\alpha\beta} = \frac{g^3 s_\theta}{16\,\pi^2}\Big[W^{WW}_0\,\delta_{\alpha\beta} + W^{WW}_1\,(Q_{+\alpha}Q_{-\beta} + Q_{+\beta}Q_{-\alpha} \\
&\quad + Q_{-\alpha}Q_{-\beta}) + W^{WW}_2\,Q_{+\alpha}Q_{+\beta}\Big].
\end{aligned}
\tag{92}
$$

This, is all what we need for a massless internal world, i.e. $m_t = 0$. Otherwise, additional contractions are needed. They are:

$$
\begin{aligned}
V^{W\phi}_\beta &= \frac{i g^3 s_\theta}{16\,\pi^2 M}\Big[W^{W\phi}_1\,Q_{+\beta} + W^{W\phi}_2\,Q_{-\beta}\Big], \\
V^{\phi W}_\alpha &= \frac{i g^3 s_\theta}{16\,\pi^2 M}\Big[W^{\phi W}_1\,Q_{+\alpha} + W^{\phi W}_2\,Q_{-\alpha}\Big], \\
V^{\phi\phi} &= \frac{g^3 s_\theta}{16\,\pi^2 M^2}\,W^{\phi\phi}.
\end{aligned}
\tag{93}
$$

24

The Ward identity that we want to prove can be written as follows:

$$\text{WI} = Q_-^\mu \sum_{i=1,4} D_\mu^i + D_V^1, \tag{94}$$

where the D^i are the four diagrams of t-channel with a photon line and the electron line removed and D_V^1 is D^1 with the inclusion of the all one-loop vertices and with the saturation already performed.

The sum of all diagrams, needed for our Ward identity, is as follows:

$$
\begin{aligned}
D_\mu^1 &= \left(\frac{ig}{2\sqrt{2}}\right)^2 gs_\theta \Big[\bar{v}\left(p_+\right) S_{l\beta}^V(1)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(1)v\left(\overline{k}\right) V_{WW\mu\lambda\rho}^0 \bar{\Delta}_{WW}^{\alpha\lambda}(s)\bar{\Delta}_{WW}^{\beta\rho}(t) \\
&+ \bar{v}\left(p_+\right) S_{l\beta}^V(1)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(1)v\left(\overline{k}\right) V_{WW\mu\lambda\rho}^0 \bar{\Delta}_{W\phi}^{\lambda}(s)\bar{\Delta}_{WW}^{\beta\rho}(t) \\
&+ \bar{v}\left(p_+\right) S_{l\beta}^V(1)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(1)v\left(\overline{k}\right) V_{W\phi\mu\rho}^0 \bar{\Delta}_{\phi W}^{\alpha}(s)\bar{\Delta}_{WW}^{\beta\rho}(t) \\
&+ \bar{v}\left(p_+\right) S_{l\beta}^V(1)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(1)v\left(\overline{k}\right) V_{W\phi\mu\rho}^0 \bar{\Delta}_{\phi\phi}(s)\bar{\Delta}_{WW}^{\beta\rho}(t) \\
&+ \bar{v}\left(p_+\right) S_{l\beta}^V(1)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(1)v\left(\overline{k}\right) V_{\phi W\mu\lambda}^0 \bar{\Delta}_{WW}^{\alpha\lambda}(s)\bar{\Delta}_{W\phi}^{\beta}(t) \\
&+ \bar{v}\left(p_+\right) S_{l\beta}^V(1)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(1)v\left(\overline{k}\right) V_{\phi W\mu\lambda}^0 \bar{\Delta}_{W\phi}^{\lambda}(s)\bar{\Delta}_{W\phi}^{\beta}(t) \\
&+ \bar{v}\left(p_+\right) S_{l\beta}^V(1)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(1)v\left(\overline{k}\right) V_{\phi\phi\mu}^0 \bar{\Delta}_{\phi W}^{\alpha}(s)\bar{\Delta}_{W\phi}^{\beta}(t) \\
&+ \bar{v}\left(p_+\right) S_{l\beta}^V(1)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(1)v\left(\overline{k}\right) V_{\phi\phi\mu}^0 \bar{\Delta}_{\phi\phi}(s)\bar{\Delta}_{W\phi}^{\beta}(t) \\
&+ \bar{v}\left(p_+\right) S_l^S(1)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(1)v\left(\overline{k}\right) V_{WW\mu\lambda\rho}^0 \bar{\Delta}_{WW}^{\alpha\lambda}(s)\bar{\Delta}_{\phi W}^{\rho}(t) \\
&+ \bar{v}\left(p_+\right) S_l^S(1)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(1)v\left(\overline{k}\right) V_{WW\mu\lambda\rho}^0 \bar{\Delta}_{W\phi}^{\lambda}(s)\bar{\Delta}_{\phi W}^{\rho}(t) \\
&+ \bar{v}\left(p_+\right) S_l^S(1)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(1)v\left(\overline{k}\right) V_{W\phi\mu\rho}^0 \bar{\Delta}_{\phi W}^{\alpha}(s)\bar{\Delta}_{\phi W}^{\rho}(t) \\
&+ \bar{v}\left(p_+\right) S_l^S(1)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(1)v\left(\overline{k}\right) V_{W\phi\mu\rho}^0 \bar{\Delta}_{\phi\phi}(s)\bar{\Delta}_{\phi W}^{\rho}(t) \\
&+ \bar{v}\left(p_+\right) S_l^S(1)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(1)v\left(\overline{k}\right) V_{\phi W\mu\lambda}^0 \bar{\Delta}_{WW}^{\alpha\lambda}(s)\bar{\Delta}_{\phi\phi}(t) \\
&+ \bar{v}\left(p_+\right) S_l^S(1)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(1)v\left(\overline{k}\right) V_{\phi W\mu\lambda}^0 \bar{\Delta}_{W\phi}^{\lambda}(s)\bar{\Delta}_{\phi\phi}(t) \\
&+ \bar{v}\left(p_+\right) S_l^S(1)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(1)v\left(\overline{k}\right) V_{\phi\phi\mu}^0 \bar{\Delta}_{\phi W}^{\alpha}(s)\bar{\Delta}_{\phi\phi}(t) \\
&+ \bar{v}\left(p_+\right) S_l^S(1)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(1)v\left(\overline{k}\right) V_{\phi\phi\mu}^0 \bar{\Delta}_{\phi\phi}(s)\bar{\Delta}_{\phi\phi}(t) \Big] \tag{95}
\end{aligned}
$$

$$
\begin{aligned}
D_\mu^2 &= \left(\frac{ig}{2\sqrt{2}}\right)^2 (-igs_\theta) \Big[\bar{v}\left(p_+\right) S_{l\mu\beta}^V(2)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(2)v\left(\overline{k}\right) \bar{\Delta}_{WW}^{\beta\alpha}(s) \\
&+ \bar{v}\left(p_+\right) S_{l\mu\beta}^V(2)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(2)v\left(\overline{k}\right) \bar{\Delta}_{W\phi}^{\beta}(s) \\
&+ \bar{v}\left(p_+\right) S_{l\mu}^S(s)v\left(q_+\right) \bar{u}\left(k\right) S_{q\alpha}^V(2)v\left(\overline{k}\right) \bar{\Delta}_{\phi W}^{\alpha}(s) \\
&+ \bar{v}\left(p_+\right) S_{l\mu}^S(s)v\left(q_+\right) \bar{u}\left(k\right) S_q^S(2)v\left(\overline{k}\right) \bar{\Delta}_{\phi\phi}(s) \Big] \tag{96}
\end{aligned}
$$

$$
D_\mu^3 = \left(\frac{ig}{2\sqrt{2}}\right)^2 (igQ_u s_\theta) \Big[\bar{v}\left(p_+\right) S_{l\beta}^V(3)v\left(q_+\right) \bar{u}\left(k\right) S_{q\mu\alpha}^V(3)v\left(\overline{k}\right) \bar{\Delta}_{WW}^{\beta\alpha}(t)
$$

$$+ \quad \bar{v}\,(p_+)\,S^V_{l\beta}(3)v\,(q_+)\,\bar{u}\,(k)\,S^S_{q\mu}(3)v\,(\overline{k})\,\bar{\Delta}^\beta_{W\phi}(t)$$

$$+ \quad \bar{v}\,(p_+)\,S^S_{l}(3)v\,(q_+)\,\bar{u}\,(k)\,S^V_{q\mu\alpha}(3)v\,(\overline{k})\,\bar{\Delta}^\alpha_{\phi W}(t)$$

$$+ \quad \bar{v}\,(p_+)\,S^S_{l}(3)v\,(q_+)\,\bar{u}\,(k)\,S^S_{q\mu}(3)v\,(\overline{k})\,\bar{\Delta}_{\phi\phi}(t)\Big] \tag{97}$$

$$D^4_\mu \quad = \quad \left(\frac{ig}{2\sqrt{2}}\right)^2 (igQ_d s_\theta) \left[\bar{v}\,(p_+)\,S^V_{l\beta}(4)v\,(q_+)\,\bar{u}\,(k)\,S^V_{q\mu\alpha}(4)v\,(\overline{k})\,\bar{\Delta}^{\beta\alpha}_{WW}(t)\right.$$

$$+ \quad \bar{v}\,(p_+)\,S^V_{l\beta}(4)v\,(q_+)\,\bar{u}\,(k)\,S^S_{q\mu}(4)v\,(\overline{k})\,\bar{\Delta}^\beta_{W\phi}(t)$$

$$+ \quad \bar{v}\,(p_+)\,S^s_{l}(4)v\,(q_+)\,\bar{u}\,(k)\,S^V_{q\mu\alpha}(4)v\,(\overline{k})\,\bar{\Delta}^\alpha_{\phi W}(t)$$

$$+ \quad \bar{v}\,(p_+)\,S^s_{l}(4)v\,(q_+)\,\bar{u}\,(k)\,S^S_{q\mu}(4)v\,(\overline{k})\,\bar{\Delta}_{\phi\phi}(t)\Big] \tag{98}$$

Here, the argument s or t for propagators denotes the variable s' and κ_+, of Eq.(85), respectively. Furthermore,

$$Q_\pm = p_\pm - q_\pm, \qquad Q = Q_+ + Q_-. \tag{99}$$

The strings of gamma-matrices are given by

$$S^V_{l\mu}(1) \quad = \quad \gamma_\mu\gamma_+, \quad S^S_l(1) = \frac{m_e}{M}\gamma_+, \quad S^V_{q\mu}(1) = \gamma_\mu\gamma_+,$$

$$S^S_q(1) \quad = \quad \frac{1}{2}\Big[\frac{m_u - m_d}{M}(\gamma_+ + \gamma_-) + \frac{m_u + m_d}{M}(\gamma_+ - \gamma_-)\Big]. \tag{100}$$

$$S^V_{l\mu\beta}(2) \quad = \quad \gamma_\mu\frac{i(\slashed{q}_+ + \slashed{Q}) + m_e}{(q_+ + Q)^2 + m_e^2}\gamma_\beta\gamma_+,$$

$$S^S_{l\mu}(2) \quad = \quad \gamma_\mu\frac{i(\slashed{q}_+ + \slashed{Q}) + m_e}{(q_+ + Q)^2 + m_e^2}\frac{m_e}{M}\gamma_+, \quad S^V_{q\mu}(2) = \gamma_\mu\gamma_+,$$

$$S^S_q(2) \quad = \quad \frac{1}{2}\Big[\frac{m_u - m_d}{M}(\gamma_+ + \gamma_-) + \frac{m_u + m_d}{M}(\gamma_+ - \gamma_-)\Big]. \tag{101}$$

$$S^V_{l\mu}(3) \quad = \quad \gamma_\mu\gamma_+, \quad S^S_l(3) = \frac{m_e}{M}\gamma_+,$$

$$S^V_{q\mu\alpha}(3) \quad = \quad \gamma_\mu\frac{i(\slashed{Q}_- - \slashed{k}) + m_u}{(Q_- - k)^2 + m_u^2}\gamma_\alpha\gamma_+,$$

$$S^S_{q\mu}(3) \quad = \quad \gamma_\mu\frac{i(\slashed{Q}_- - \slashed{k}) + m_u}{(Q_- - k)^2 + m_u^2}(\frac{m_u}{M}\gamma_+ - \frac{m_d}{M}\gamma_-). \tag{102}$$

$$S^V_{l\mu}(4) \quad = \quad \gamma_\mu\gamma_+, \quad S^S_l(4) = \frac{m_e}{M}\gamma_+,$$

$$S^V_{q\mu\alpha}(4) \quad = \quad \gamma_\alpha\gamma_+\frac{-i(\slashed{Q}_- - \overline{\slashed{k}}) + m_d}{(Q_- - \overline{k})^2 + m_d^2}\gamma_\mu,$$

$$S^S_{q\mu}(4) \quad = \quad (\frac{m_u}{M}\gamma_+ - \frac{m_d}{M}\gamma_-)\frac{-i(\slashed{Q}_- - \overline{\slashed{k}}) + m_d}{(Q_- - \overline{k})^2 + m_d^2}\gamma_\mu. \tag{103}$$

The remaining tree-level vertices, with gs_θ factorized out, are given by

$$V^0_{W\phi\mu\alpha} = iM\delta_{\mu\alpha}, \quad V^0_{\phi W\mu\alpha} = -iM\delta_{\mu\alpha}, \quad V^0_{\phi\phi\mu} = Q_{+\mu} + Q_\mu. \tag{104}$$

The D^1_V one-loop contraction of Eq.(94) is obtained from D^1_μ of Eq.(95) after multiplication by $Q_{-\mu}$ and after replacing the lowest order triple vertices ($\gamma W^+ W^-$, $\gamma W^+ \phi^-$, $\gamma \phi^+ W^-$ and $\gamma \phi^+ \phi^-$) with the one-loop corrected ones. The corresponding contractions are given in Eqs.(92–93).

5.1 The massless internal world.

To prove the Ward identity is particularly simple if we neglect all fermion masses in loops. To make the final result more compact we introduce the following notations:

$$\begin{align}
\Gamma^\pm_l(\mu) &= \bar{v}(p_+)\gamma_\mu\gamma_\pm v(q_+), \\
\Gamma^\pm_q(\mu) &= \bar{u}(k)\gamma_\mu\gamma_\pm v(\overline{k}), \\
\Gamma^\pm_l &= \bar{v}(p_+)\gamma_\pm v(q_+), \\
\Gamma^\pm_q &= \bar{u}(k)\gamma_\pm v(\overline{k}),
\end{align} \tag{105}$$

and also,

$$\begin{align}
\Gamma(Q_-,+,1,\pm) &= m_u\,\Gamma^+_l(Q_-)\,\Gamma^+_q - m_d\,\Gamma^+_l(Q_-)\,\Gamma^-_q, \\
\Gamma(1,+,1,\pm) &= m_u\,\Gamma^+_l\,\Gamma^+_q - m_d\,\Gamma^+_l\,\Gamma^-_q.
\end{align} \tag{106}$$

For the non-vertex part of the Ward identity, after using the Dirac equation, we get the following result:

$$\frac{1}{gs_\theta}Q^\mu_- \sum_{i=1,4} D^i_\mu = \mathrm{WI}_{\mathrm{nv}}, \tag{107}$$

where, with $\mu^2_y = M^2/ys$, the contractions is

$$\begin{align}
\mathrm{WI}_{\mathrm{nv}} &= \frac{i\,g^2}{16}\Gamma(Q_-,+,1,\pm)\frac{R_s}{x_2 - \mu^2_y}\frac{1}{P_s}\left(\frac{1}{ys} + \frac{Y - x_2}{P_t}\right), \\
&+ \frac{g^2 m_e}{16}\Gamma(1,+,1,\pm)\frac{R_s R_t}{P_s P_t}\left[\frac{ys}{M^2}(x_2 - Y) + \frac{1}{2}\frac{x_2 + X - Y}{x_2 - \mu^2_y}\right. \\
&+ \left.\frac{1}{2}\frac{x_2 - X - Y}{Y - \mu^2_y}\right], \\
&+ \frac{g^2 m_e}{16\,M^2}\Gamma(1,+,1,\pm)\frac{R_s}{P_s}\left[\frac{ys}{P_t}(x_2 - Y) - 1\right], \\
&+ \frac{g^2 m_e}{16\,M^2}\Gamma(1,+,1,\pm)\frac{R_t}{P_t}\left[\frac{ys}{P_s}(x_2 - Y) + 1\right], \\
&+ \frac{g^2 m_e}{16\,M^2}\Gamma(1,+,1,\pm)\left[\frac{x_2 - Y}{P_s P_t} + \frac{1}{P_t} - \frac{1}{P_s}\right]
\end{align}$$

27

$$+ \quad \frac{i\,g^2 m_e}{16}\,\Gamma\,(1,+,Q_-,+)\,\frac{R_t}{Y-\mu_y^2}\frac{1}{P_t}\left[\frac{Y-x_2}{P_s}-1\right],$$

$$+ \quad \frac{g^2}{16}\,\Gamma_l^+(\mu)\,\Gamma_q^+(\mu)\left[\frac{ys}{P_s P_t}\,(x_2-Y)+\frac{1}{P_t}-\frac{1}{P_s}\right]. \tag{108}$$

The propagators are expressed as

$$P_s = \left[-s'+M^2-S_W^0(s')\right]^{-1}, \qquad P_t = \left[-\kappa_+ +M^2-S_W^0(\kappa_+)\right]^{-1}. \tag{109}$$

Moreover we have ratios of bare to running masses,

$$R(p^2) = \frac{M^2}{M^2(p^2)}-1, \qquad R_s = R(s'), \quad R_t = R(\kappa_+). \tag{110}$$

This case is relatively simple and, therefore, we give the derivation in some details. The expression that we have obtained is still written in terms of bare quantities and renormalization can be achieved by using the following set of relations:

$$S_\phi(p^2) = 0, \qquad S_W^0(p^2) = p^2\left[1-\frac{g^2}{g^2(p^2)}\right],$$

$$M^2 = M^2(p^2)\frac{g^2}{g^2(p^2)} = \frac{g^2}{g^2(p_W)}\,p_W, \qquad R(p^2) = \frac{g^2}{g^2(p^2)}-1,$$

$$p^2+M^2 = \frac{g^2}{g^2(p_W)}\,p_W\left\{\left[1+\frac{g^2(p_W)}{16\,\pi^2}\,\Pi_{3Q}(p_W)\right]\frac{p^2}{p_W}+1\right\} \tag{111}$$

Furthermore, the propagators are rewritten as

$$P_s^{-1} = -\frac{G_s}{g^2}\frac{1}{p_s}, \qquad p_s = x_2 ys - p_W + \left(1-\frac{G_s}{G_W}\right)p_W,$$

$$P_t^{-1} = -\frac{G_t}{g^2}\frac{1}{p_t}, \qquad p_t = Y ys - p_W + \left(1-\frac{G_t}{G_W}\right)p_W, \tag{112}$$

where $G_W = g^2(p_W)$. We introduce an auxiliary function

$$\Phi_W(x) = \left[\left(1+\frac{G_W}{16\,\pi^2}\,\Pi_W\right)\frac{xys}{p_W}-1\right]^{-1}, \qquad \Pi_W = \Pi_{3Q}(p_W). \tag{113}$$

In this way, the non-vertex part of the Ward identity will become

$$\frac{p_s p_t}{G_s G_t}\,\mathrm{WI}_{\mathrm{nv}} = \frac{i}{16}\,\Gamma\,(Q_-,+,1,\pm)\,\frac{\Pi_s}{16\,\pi^2}$$

$$\times \left[1-\frac{Y ys}{p_W}\frac{\Pi_t}{16\,\pi^2}\,G_W\,\Phi_W(x_2)\right]$$

$$+ \frac{im_e}{16}\,\Gamma\,(1,+,1,\pm)\left\{\frac{\Pi_s-\Pi_t}{16\,\pi^2}+G_W\,\frac{\Pi_s\Pi_t}{256\,\pi^4}\frac{ys}{2\,p_W}\right.$$

28

$$\times \ \Big[(x_2 + X - y)\,\Phi_w(x_2) + (x_2 - X - Y)\,\Phi_w(Y)\Big]\Big\}$$

$$+ \ \ \frac{im_e}{16}\,\Gamma_l^+\,\Gamma_q^+(Q_-)\,\frac{\Pi_t}{16\,\pi^2}\,\Big[\frac{x_2 ys}{p_w}\,\frac{\Pi_s}{16\,\pi^2}\,\Phi_w(Y) - 1\Big]$$

$$+ \ \ \Gamma_l^+(\mu)\,\Gamma_q^+(\mu)\,\frac{x_2\Pi_s - Y\Pi_t}{16\,\pi^2}. \tag{114}$$

As usual, we have introduced subscripts s, t to denote $\Pi_s = \Pi_{3Q}(s')$ and $\Pi_t = \Pi_{3Q}(\kappa_+)$. Also for the vertices we get a contribution to the Ward identity that we write as

$$\frac{1}{gs_\theta}\,D_V^1 = \mathrm{WI_v}. \tag{115}$$

Two additional quantities are needed,

$$\mu_y^2 = \frac{M^2}{ys}, \qquad \Phi(x) = (x - \mu_y^2)^{-1}. \tag{116}$$

The function Φ is related to Φ_w by

$$\Phi(x) \ \ = \ \ \frac{G_w}{g^2}\,\frac{ys}{p_w}\,\Phi_w(x) = (\Phi_w + 1)\,\frac{\Phi_w(x)}{\Phi_w},$$
$$\Phi_w \ \ \equiv \ \ \Phi_w(1). \tag{117}$$

The explicit expression for this part of the Ward identity is

$$16\,\pi^2\,\frac{p_s p_t}{G_s G_t}\,\mathrm{WI_v} = \mathrm{WI'_v}, \tag{118}$$

$$\mathrm{Wi'_v} \ \ = \ \ \frac{i}{16}\,\Gamma\,(Q_-, +, 1, \pm)\,\Big[R_s\,\Phi(x_2)\,\Big(\frac{1}{ys}W_0^{ww} - x_2\,W_1^{ww}\Big) - W_1^{ww}\Big]$$

$$+ \ \ \frac{m_e}{32}\,\Gamma\,(1, +, 1, \pm)\,\Big\{R_s R_t\,\Phi(x_2)\,\Phi(Y)\Big[\frac{2\,x_2 + X}{ys}\,W_0^{ww} - x_2\,(X + 2\,Y)\,W_2^{ww}\Big]$$

$$+ \ \ R_s R_t\,\Phi(x_2)\Big[\frac{1}{ys}\,W_0^{ww} + (x_2 + X + Y)\,W_1^{ww} - (X + Y)\,W_2^{ww}\Big]$$

$$+ \ \ R_s R_t\,\Phi(Y)\Big[-\frac{1}{ys}\,W_0^{ww} - (x_2 + X + Y)\,W_1^{ww} + (X + Y)\,W_2^{ww}\Big]$$

$$+ \ \ R_s\,\Phi(x_2)\Big[\frac{2}{ys}\,W_0^{ww} + (-x_2 + X + Y)\,W_1^{ww} - (x_2 + X + Y)\,W_2^{ww}\Big]$$

$$+ \ \ R_t\,\Phi(Y)\Big[\frac{2}{ys}\,W_0^{ww} + (-x_2 - X + Y)\,W_1^{ww} - 2\,Y\,W_2^{ww}\Big] - 2\,W_2^{ww}\Big\}$$

$$+ \ \ \frac{im_e}{16}\,\Gamma_l^+\,\Gamma_q^+(Q_-)\,\Big\{R_t\,\Phi(Y)\Big[\frac{1}{ys}\,W_0^{ww} + Y\,W_1^{ww} - Y\,W_2^{ww}\Big]$$

$$+ \ \ W_1^{ww} - W_2^{ww}\Big\} - \frac{1}{16}\,\Gamma_l^+(\mu)\,\Gamma_q^+(\mu)\,W_0^{ww} \tag{119}$$

29

To go one step further, to this expression we apply the renormalization procedure of Eqs.(111–112). By inspection, its is seen that the Ward identity

$$\mathrm{WI}_{\mathrm{nv}} + \mathrm{WI}_{\mathrm{v}} = 0, \tag{120}$$

is satisfied if the following conditions hold:

$$
\begin{aligned}
W_0^{ww} &= (x_2\,\Pi_s - Y\,\Pi_t)\,ys, \\
W_1^{ww} &= \Pi_s, \\
W_2^{ww} &= \Pi_s - \Pi_t. \tag{121}
\end{aligned}
$$

An explicit calculation shows that

$$
\begin{aligned}
\Pi_s &= \frac{4}{9} - \frac{4}{3}\,B_0\left(-s';0,0\right), \\
\Pi_t &= \frac{4}{9} - \frac{4}{3}\,B_0\left(-\kappa_+;0,0\right), \\
W_0^{ww} &= \left\{\frac{4}{9}\,(x_2 - Y) + \frac{4}{3}\left[Y\,B_0\left(-\kappa_+;0,0\right) - x_2\,B_0\left(-s';0,0\right)\right]\right\}, \\
W_1^{ww} &= \frac{4}{9} - \frac{4}{3}\,B_0\left(-s';0,0\right), \\
W_2^{ww} &= \frac{4}{3}\left[B_0\left(-\kappa_+;0,0\right) - B_0\left(-s';0,0\right)\right], \tag{122}
\end{aligned}
$$

which represents the solution of Eq.(120).

5.2   The case of non-zero m_t.

When the internal world is not massless the derivation of the Ward identity, although straightforward, is considerably lengthy. First, we have to change the normalization equations, Eqs.(111–112). Now they can be written as

$$
\begin{aligned}
P_s^{-1} &= -\frac{G_s}{g^2}\frac{1}{p_s}, \quad p_s = x_2 ys - p_w + \left(1 - \frac{G_s}{G_w}\right)p_w + G_s\left[f_w(s') - f_w(p_w)\right], \\
P_t^{-1} &= -\frac{G_t}{g^2}\frac{1}{p_t}, \quad p_t = Y ys - p_w + \left(1 - \frac{G_t}{G_w}\right)p_w + G_t\left[f_w(\kappa_+) - f_w(p_w)\right], \tag{123}
\end{aligned}
$$

Furthermore, the relations between bare and running quantities is now modified into the following form:

$$
\begin{aligned}
\frac{p_w}{M^2(p^2)} &= \frac{g^2(p_w)}{g^2(p^2)}\,\eta(p^2), \\
\eta(p^2) &= \left\{1 - \frac{g^2(p^2)}{16\,\pi^2}\left[\sigma_w(p^2) - \frac{m_t^2}{p^2}\,\sigma_\phi(p^2)\right]\right\} \\
&\quad\times \left\{1 - \frac{g^2(p_w)}{16\,\pi^2}\left[\sigma_w(p_w) + \frac{m_t^2}{p_w}\,\sigma_\phi(p^2)\right]\right\}^{-1},
\end{aligned}
$$

$$M^2 = M^2(p^2) \frac{g^2}{g^2(p^2)} \eta(p^2) + \frac{g^2}{16\,\pi^2}\, f_w(p_w),$$

$$R(p^2) = \frac{g^2}{g^2(p^2)} \eta(p^2) \left[1 - \frac{g^2(p_w)}{16\,\pi^2}\, \sigma_w(p_w)\right] - 1. \tag{124}$$

Moreover, all vertices contribute. We have created a FORM program that computes the massive-massive Ward identity [3] and have found that the latter is satisfied if

$$W_0^{ww} = x_2 ys\,(\Pi_s + \sigma_w^s) - Yys\,(\Pi_t + \sigma_w^t),$$

$$W_1^{ww} = \Pi_s + \sigma_s^W + \frac{m_t^2}{x_2 ys}\,\sigma_s^\phi,$$

$$W_2^{ww} = \Pi_s - \Pi_t + \sigma_w^s - \sigma_w^t + \frac{m_t^2}{x_2 ys}\,\sigma_\phi^s - \frac{m_t^2}{Yys}\,\sigma_\phi^t,$$

$$W_1^{w\phi} = m_t^2\,(\sigma_\phi^t - \sigma_\phi^s),$$

$$W_2^{w\phi} = -m_t^2\,\sigma_\phi^s,$$

$$W^{\phi\phi} = m_t^2\,(x_2 \sigma_\phi^s - Y\sigma_\phi^t)\,ys. \tag{125}$$

We have also written a FORM program for the evaluation of vertices [4] derived from a more general one that computes all one-loop diagrams in the standard model. From it we derive

$$W_i^{\phi w} = -W_i^{w\phi}, \tag{126}$$

and, with N_g generations of fermions, the result is as follows:

$$
\begin{aligned}
W_0^{ww} =\ & N_g ys\left\{\frac{4}{9}(x_2 - Y) + \frac{4}{3}\left[YB_0\,(-\kappa_+;0,0) - x_2 B_0\,(-s';0,0)\right]\right\} \\
& + \frac{1}{3}ys(x_2 - Y) - \frac{1}{2}\frac{m_t^4}{Yys}B_0\,(-\kappa_+;0,m_t) + YysB_0\,(-\kappa_+;0,m_t) \\
& - \frac{1}{2}m_t^2 B_0\,(-\kappa_+;0,m_t) + \frac{1}{2}\frac{m_t^4}{x_2 ys}B_0\,(-s';0,m_t) \\
& - x_2 ysB_0\,(-s';0,m_t) + \frac{1}{2}m_t^2 B_0\,(-s';0,m_t) + \frac{1}{2}(\frac{1}{x_2} - \frac{1}{Y})\frac{m_t^2}{ys}A_0\,(m_t) \\
& - \frac{1}{3}ys(x_2 - Y) - YysB_0\,(-\kappa_+;0,0) + x_2 ysB_0\,(-s';0,0),
\end{aligned}
$$

$$
\begin{aligned}
W_1^{ww} =\ & N_g\left[\frac{4}{9} - \frac{4}{3}B_0\,(-s';0,0)\right] - \frac{m_t^2}{x_2 ys} + 2\left(\frac{m_t^2}{x_2 ys}\right)^2 B_0\,(-s';0,m_t) \\
& - B_0\,(-s';0,m_t)\frac{m_t^2}{x_2 ys} - B_0\,(-s';0,m_t) + 2\frac{m_t^2}{(x_2 ys)^2}A_0\,(m_t)
\end{aligned}
$$

[3] available from http://www.to.infn.it/giampier/form/mflwi.f
[4] available from http://www.to.infn.it/giampier/form/vertsw.f

$$- \frac{1}{x_2 ys} A_0\left(m_t\right) + B_0\left(-s'; 0, 0\right),$$

$$
\begin{aligned}
W_2^{ww} &= \frac{4}{3} N_g \Big[B_0\left(-\kappa_+; 0, 0\right) - B_0\left(-s'; 0, 0\right) \Big] \\
&+ \frac{m_t^2}{ys}\left(-\frac{1}{x_2} + \frac{1}{Y}\right) - 2\left(\frac{m_t^2}{Yys}\right)^2 B_0\left(-\kappa_+; 0, m_t\right) \\
&+ \frac{m_t^2}{Yys} B_0\left(-\kappa_+; 0, m_t\right) + B_0\left(-\kappa_+; 0, m_t\right) + 2\left(\frac{m_t^2}{x_2 ys}\right)^2 B_0\left(-s'; 0, m_t\right) \\
&- \frac{m_t^2}{x_2 ys} B_0\left(-s'; 0, m_t\right) - B_0\left(-s'; 0, m_t\right) + 2\frac{m_t^2}{(ys)^2}\left(\frac{1}{x_2^2} - \frac{1}{Y^2}\right) B_0\left(-s'; 0, m_t\right) \\
&+ \left(-\frac{1}{x_2 ys} + \frac{1}{Yys}\right) A_0\left(m_t\right) - B_0\left(-\kappa_+; 0, 0\right) + B_0\left(-s'; 0, 0\right),
\end{aligned}
$$

$$
\begin{aligned}
W_1^{w\phi} &= \frac{3}{2}\frac{m_t^4}{Yys} B_0\left(-\kappa_+; 0, m_t\right) - \frac{3}{2} m_t^2 B_0\left(-\kappa_+; 0, m_t\right) \\
&- \frac{3}{2}\frac{m_t^4}{x_2 ys} B_0\left(-s'; 0, m_t\right) + B_0\left(-s'; 0, m_t\right) m_t^2\left(\frac{3}{2}\right) - \frac{3}{2}\frac{m_t^2}{ys}\left(\frac{1}{x_2} - \frac{1}{Y}\right) A_0\left(m_t\right),
\end{aligned}
$$

$$
\begin{aligned}
W_2^{w\phi} &= -\frac{3}{2}\frac{m_t^4}{x_2 ys} B_0\left(-s'; 0, m_t\right) \\
&+ \frac{3}{2} m_t^2 B_0\left(-s'; 0, m_t\right) - \frac{3}{2}\frac{m_t^2}{x_2 ys} A_0\left(m_t\right),
\end{aligned}
$$

$$
\begin{aligned}
W^{\phi\phi} &= \frac{3}{2} Yys\, m_t^2 B_0\left(-\kappa_+; 0, m_t\right) - \frac{3}{2} m_t^4 B_0\left(-\kappa_+; 0, m_t\right) \\
&- \frac{3}{2} x_2 ys\, m_t^2 B_0\left(-s'; 0, m_t\right) + \frac{3}{2} m_t^4 B_0\left(-s'; 0, m_t\right)
\end{aligned}
\tag{127}
$$

The remaining quantities have already been given, but we repeat them, for completeness, in their scalarized form:

$$
\begin{aligned}
f_w(s') &= -\frac{1}{3} m_t^2 - \frac{4}{3} B_0\left(-s'; m_t, m_t\right) m_t^2 \\
&- \frac{2}{3} x_2 ys B_0\left(-s'; m_t, m_t\right) - \frac{1}{2} B_0\left(-s'; 0, m_t\right) m_t^2 - \frac{1}{2}\frac{m_t^4}{x_2 ys} B_0\left(-s'; 0, m_t\right) \\
&+ x_2 ys B_0\left(-s'; 0, m_t\right) - \frac{1}{3} x_2 ys B_0\left(-s'; 0, 0\right) - \frac{1}{2}\frac{m_t^2}{x_2 ys} A_0\left(m_t\right) - \frac{1}{3} A_0\left(m_t\right),
\end{aligned}
$$

$$
\begin{aligned}
f_w(\kappa_+) &= -\frac{1}{3} m_t^2 - \frac{4}{3} B_0\left(-\kappa_+; m_t, m_t\right) m_t^2 \\
&- \frac{2}{3} Yys B_0\left(-\kappa_+; m_t, m_t\right) - \frac{1}{2} B_0\left(-\kappa_+; 0, m_t\right) m_t^2 - \frac{1}{2}\frac{m_t^4}{Yys} B_0\left(-\kappa_+; 0, m_t\right) \\
&+ Yys B_0\left(-\kappa_+; 0, m_t\right) - \frac{1}{3} Yys B_0\left(-\kappa_+; 0, 0\right) - \frac{1}{2}\frac{m_t^2}{Yys} A_0\left(m_t\right) - \frac{1}{3} A_0\left(m_t\right).
\end{aligned}
$$

32

Furthermore, we have

$$\sigma_\phi^s = \frac{3}{2}\frac{1}{x_2 ys}\left[A_0\left(m_t\right) + \left(m_t^2 - x_2 ys\right)B_0\left(-s';0,m_t\right)\right],$$

$$\sigma_\phi^t = \frac{3}{2}\frac{1}{Yys}\left[A_0\left(m_t\right) + \left(m_t^2 - Yys\right)B_0\left(-\kappa_+;0,m_t\right)\right]. \qquad (129)$$

The explicit expressions shown in Eqs.(127–129) are solutions of Eq.(125), therefore proving the validity of the U(1) massive-massive Ward identity. The proof is straightforward, although the FORM program that derives WI = 0 generates of the order of 100,000 in some intermediate step.

5.3 Fixed-width scheme

In the previous section we have shown that the massive-massive Ward identity is satisfied in the Fermion-Loop scheme when vertices are added. In many numerical evaluations, where one would like to save as much as possible of CPU time, the implementation of the Fermion-Loop is very time-consuming. Therefore, one would like to have a gauge-preserving scheme which is as simple as possible from the point of view of building an event generator. For conserved currents this scheme is given by the use of a fixed width for vector bosons in all channel, not only annihilation but also scattering. It is a nonsense, from the point of view of a full-fledged field-theoretical formulation, but it has been shown [2] that the numerical differences, with respect to the Fermion-Loop scheme, are tiny at LEP 2 energies and also for small, but not vanishing, electron's scattering angle.

We have taken the complete, massive-massive, Ward identity and have neglected all vertex corrections. In this case one can show that the Ward identity is satisfied with the following choice:

$$\begin{aligned}
P_s^{-1} &= -x_2 ys + M^2 - i\,M\Gamma_W,\\
P_t^{-1} &= -Yys + M^2 - i\,M\Gamma_W,\\
M^2(p^2) &= M^2, \quad \text{for} \quad p^2 = -s' \quad \text{and} \quad p^2 = -\kappa_+,
\end{aligned} \qquad (130)$$

where Γ_W is the W, on-shell, total width. Therefore, we have a formulation of the fixed-width scheme that works also for non-zero external fermion masses.

6 Approximations.

A full implementation of the Fermion-Loop scheme can be achieved without having to rely on any sort of approximation. Of course, one has to compute all form-factors occurring in the vertices and this is, notoriously, a lengthy procedure.

One of the beautiful aspects of computing the Ward identity is that, in the final answer for vertices, all Gram determinants disappear and a full scalarization of the answer has been successfully achieved. This is not the case when we need the vertices for the amplitude, since all powers of Gram determinants, up to the third one, will appear.

For this reason we start investigating into the numerical impact of mass corrections. Having proved the relevant Ward identity for the fully massive case, we have been able to show that masses do not spoil any cancellation mechanism. In other words, the $1+5+4$ diagrams in the t-channel preserve gauge invariance in the Fermion-Loop scheme with all masses kept explicitly. Therefore, inside this set of diagrams we are allowed to investigate the numerical relevance of masses and to neglect some, if convenient.

The dominant effect is due to the collinear region where the mass in the electron line acts as a regulator,

$$\frac{1}{4}\, L_{\mu\nu} W^{\mu\nu} = W_1 \left[2\, \frac{m_e^2}{(Xys)^2} - \frac{1}{Xys} \right] + 2\, W_2\, \frac{y-1}{Xy^3 s} + \mathcal{O}\left(X \right). \qquad (131)$$

It is clear that terms proportional to m_e can be safely neglected inside $W_{1,2}$ and, therefore, we can take the incoming positron as massless without effecting the numerical precision of the result. The quark masses are only relevant, from a numerical point of view, for the multi-peripheral diagrams. In the amplitude we must keep all terms proportional to

$$\frac{1}{(Q_- - k)^2 + m_u^2} \qquad \frac{m_u^2}{\left[(Q_- - k)^2 + m_u^2 \right]^2}, \qquad (132)$$

and similar ones for the down quark. Once we neglect m_e, apart from the photon flux-function, the only terms proportional to the quark masses arise from the internal propagators of the u, d-quarks and not from couplings. Propagators in the multi-peripheral diagrams are, therefore, the only place where we would like to keep quark masses. But one can easily see that, in the complete Ward identity, all terms proportional to the quark propagators drop out, therefore, we can write the Ward identity for $m_e = m_u = m_d = 0$. In this case we derive

$$\Gamma_l^P(\mu)\, \Gamma_q^P(\mu) \left[x_2\, \frac{\Pi_s}{16\,\pi^2} - Y\, \frac{\Pi_t}{16\,\pi^2} \right], \qquad (133)$$

from transitions, and

$$- \Gamma_l^P(\mu)\, \Gamma_q^P(\mu)\, \frac{W_0^{WW}}{16\,\pi^2}, \qquad (134)$$

from vertices, that indeed add up to zero.

To summarize, the Fermion-Loop scheme with all external fermion masses set to zero, but the electron mass in the photon flux-function and the quark masses in the quark propagators of the multi-peripheral diagrams, preserves gauge invariance.

From a numerical point of view, this is all we need in evaluating the single-W process. Using the Fermion-Loop, as a gauge preserving scheme, has, moreover, the bonus of automatically setting the correct scale for all running coupling constants.

7 Cancellation of the ultraviolet divergences.

When we assume that the W-currents are treated as conserved ones, then ten operators are enough to describe the γWW vertex, even for proving the U(1) Ward identity. We will use

$$
\begin{aligned}
W^1_{\mu\alpha\beta} &= 4\mathcal{N}\left[\delta_{\alpha\beta}\,Q_{+\mu} + \delta_{\mu\beta}\,Q_{-\alpha} - \delta_{\mu\alpha}\,Q_\beta\right],\\
W^{2,3}_{\mu\alpha\beta} &= 2\mathcal{N}\left[\delta_{\mu\beta}\,Q_{-\alpha} \mp \delta_{\mu\alpha}\,Q_\beta\right],\\
W^4_{\mu\alpha\beta} &= 2/,\mathcal{N}^3\,Q_{+\mu}\,Q_{-\alpha}\,Q_\beta,\\
W^5_{\mu\alpha\beta} &= \mathcal{N}\left[\varepsilon(Q_+,\mu,\alpha,\beta) - \varepsilon(Q_-,\mu,\alpha,\beta)\right],\\
W^6_{\mu\alpha\beta} &= -\mathcal{N}\left[\varepsilon(Q_+,\mu,\alpha,\beta) + \varepsilon(Q_-,\mu,\alpha,\beta)\right],\\
W^7_{\mu\alpha\beta} &= -\mathcal{N}^3\,\varepsilon(Q_+,Q_-,\alpha,\beta)\,Q_\mu,\\
W^8_{\mu\alpha\beta} &= 2\mathcal{N}^3\,Q_{-\mu}\,Q_{-\alpha}\,Q_\beta,\\
W^9_{\mu\alpha\beta} &= 2\mathcal{N}\,\delta_{\alpha\beta}\,Q_{-\mu},\\
W^{10}_{\mu\alpha\beta} &= \mathcal{N}^3\,\varepsilon(Q_+,Q_-,\alpha,\beta)\,Q_{+\mu}.
\end{aligned}
\tag{135}
$$

We will write the vertex as

$$
W^{\gamma WW}_{\mu\alpha\beta} = \frac{g^3 s_\theta}{16\,\pi^2}\,(Xys)^{1/2}\sum_{i=1,10} I_i^{\gamma WW}\,W^i_{\mu\alpha\beta}.
\tag{136}
$$

For one generation, l, ν_l, u, d, of massless fermions we obtain

$$
\begin{aligned}
I^1 ={}& x_2\,y\,s\,(-C_{32} - C_{33} + 2\,C_{34})\\
&+ X\,y\,s\,(C_{12} + C_{22} - C_{33} + C_{34}) + Y\,y\,s\,(C_{21} + C_{22} - 2\,C_{23} + C_{31}\\
&- 2\,C_{33} + C_{34}) - \frac{2}{3} + 2\,C_{24} - 2\,C_{35} + 2\,C_{36},
\end{aligned}
$$

$$
\begin{aligned}
I^2 ={}& x_2\,y\,s\,(-2\,C_{22} + 2\,C_{23} + 3\,C_{32} + 3\,C_{33} - 6\,C_{34})\\
&+ X\,y\,s\,(-C_{12} - 3\,C_{22} + 2\,C_{23} + 3\,C_{33} - 3\,C_{34})\\
&+ Y\,y\,s\,(-2\,C_{11} + 2\,C_{12} - 5\,C_{21} - 3\,C_{22} + 8\,C_{23})\\
&- 3\,C_{31} + 6\,C_{33} - 3\,C_{34} + 2\,C_{24} + 6\,C_{35} - 6\,C_{36},
\end{aligned}
$$

$$
\begin{aligned}
I^3 ={}& x_2\,y\,s\,(-2\,C_{22} + 2\,C_{23} - C_{32} + C_{33})\\
&+ X\,y\,s\,(C_{12} + C_{22} + 2\,C_{23} + C_{33} + C_{34}) + Y\,y\,s\,(-C_{21} + C_{22}
\end{aligned}
$$

$$- C_{31} + C_{34}) + 2\,C_{24} + 2\,C_{35} + 2\,C_{36},$$

$$I^4 = 8\,X\,y\,s\,(-C_{22} + C_{23} + C_{33} - C_{34}),$$

$$I^{5,6,7} = 0,$$

$$I^8 = X\,y\,s\,(-4\,C_{12} - 8\,C_{22} - 4\,C_{23} - 8\,C_{34}),$$

$$\begin{aligned}
I^9 &= x_2\,y\,s\,(2\,C_{12} + 2\,C_{23} - 2\,C_{32} + 2\,C_{34}) \\
&+ X\,y\,s\,(2\,C_{12} + 2\,C_{22} + 2\,C_{23} + 2\,C_{34}) \\
&+ Y\,y\,s\,(-2\,C_{11} - 2\,C_{21} + 2\,C_{22} - 2\,C_{23} - 2\,C_{33} + 2\,C_{34}) \\
&- \frac{2}{3} + 4\,C_{24} + 4\,C_{36}
\end{aligned}$$

$$I^{10} = 0, \tag{137}$$

where all C_{ij}-functions are computed with zero internal masses. If we compute the ultraviolet divergent part of the vertex, for one massless generation, then the following results will emerge:

$$W_{\mu\alpha\beta}^{\gamma W W}\,|_{\rm UV} = \frac{g^3 s_\theta}{16\,\pi^2}\,(Xys)^{1/2}\,\frac{2}{3}\,\frac{1}{\bar{\varepsilon}}\left[W_{\mu\alpha\beta}^1 + W_{\mu\alpha\beta}^9\right], \tag{138}$$

but, for conserved currents, i.e. $Q_{+\beta} = 0, Q_{+\alpha} = -Q_{-\alpha}$, we get

$$W_{\mu\alpha\beta}^1 + W_{\mu\alpha\beta}^9 = 2\,\mathcal{N}\,V_{\mu\alpha\beta}^0. \tag{139}$$

Therefore, the ultraviolet divergent part of the vertex is proportional to the lowest order. This has an important consequence, in view of the fact that inside Eq.(30) there is a remaining factor $1/g^2$. The combination of bare quantities and of one-loop corrections,

$$\frac{1}{g^2} + \frac{1}{8\,\pi^2}\,(I_1 + I_9)\,|_{\rm UV}, \tag{140}$$

appearing in the total amplitude, is ultraviolet finite.

For the $t - b$ doublet result we need two different configurations of internal masses, $(m_t, 0, m_t)$ and $(0, m_t, 0)$. With

$$C_{ij}\,(t) = C_{ij}\,(0, m_t, 0) \qquad C_{ij}\,(tt) = C_{ij}\,(m_t, 0, m_t), \tag{141}$$

and

$$C_{ij}^+ = C_{ij}\,(tt) + \frac{1}{2}\,C_{ij}\,(t), \qquad C_{ij}^- = C_{ij}\,(tt) - \frac{1}{2}\,C_{ij}\,(t), \tag{142}$$

the resulting expressions for the coefficients are as follows:

$$I_1 = x_2 y s\left[-\frac{1}{2}C_{32}^+ - \frac{1}{2}C_{33}^+ + C_{34}^+\right]$$

$$+ \quad \frac{1}{2}Xys\left[C_{12}^+ + C_{22}^+ - C_{33}^+ + C_{34}^+\right]$$

$$+ \quad Yys\left[\frac{1}{2}C_{21}^+ + \frac{1}{2}C_{22}^+ - C_{23}^+ + \frac{1}{2}C_{31}^+ - C_{33}^+ + \frac{1}{2}C_{34}^+\right]$$

$$+ \quad \frac{1}{2}m_t^2\left[C_{11}(tt) - C_{12}(tt)\right] + \frac{1}{2}m_t^2 C_0(tt)$$

$$- \quad \frac{1}{2} + C_{24}^+ - C_{35}^+ + C_{36}^+,$$

$$
\begin{aligned}
I_2 \;=\; & x_2 ys\left[-C_{22}^+ + C_{23}^+ + \frac{3}{2}C_{32}^+ + \frac{3}{2}C_{33}^+ - 3C_{34}^+\right] \\
+ \; & Xys\left[-\frac{1}{2}C_{12}^+ - \frac{3}{2}C_{22}^+ + C_{23}^+ + \frac{3}{2}C_{33}^+ - \frac{3}{2}C_{34}^+\right] \\
+ \; & Yys\left[-C_{11}^+ + C_{12}^+ - \frac{5}{2}C_{21}^+ - \frac{3}{2}C_{22}^+\right. \\
+ \; & \left.4C_{23}^+ - \frac{3}{2}C_{31}^+ + 3C_{33}^+ - \frac{3}{2}C_{34}^+\right] \\
- \; & \frac{1}{2}m_t^2\left[C_{11}(tt) - C_{12}(tt)\right] - \frac{1}{2}m_t^2 C_0(tt) + C_{24}^+ + 3C_{35}^+ - 3C_{36}^+
\end{aligned}
$$

$$
\begin{aligned}
I_3 \;=\; & x_2 ys\left[-C_{22}^+ + C_{23}^+ - \frac{1}{2}C_{32}^+ + \frac{1}{2}C_{33}^+\right] \\
+ \; & Xys\left[\frac{1}{2}C_{12}^+ + \frac{1}{2}C_{22}^+ + C_{23}^+ + \frac{1}{2}C_{33}^+ + \frac{1}{2}C_{34}^+\right] \\
+ \; & \frac{1}{2}Yys\left[-C_{21}^+ + C_{22}^+ - C_{31}^+ + C_{34}^+\right] + \frac{1}{2}m_t^2\left[C_{11}(tt) + C_{12}(tt)\right] \\
+ \; & \frac{1}{2}m_t^2 C_0(tt) + C_{24}^+ + C_{35}^+ + C_{36}^+
\end{aligned}
$$

$$I_4 \;=\; 4Xys\left[-C_{22}^+ + C_{23}^+ + C_{33}^+ - C_{34}^+\right]$$

$$
\begin{aligned}
I_5 \;=\; & x_2 ys\left[-C_{32}^- + C_{33}^-\right] + Xys\left[C_{12}^- + C_{22}^- + 2C_{23}^- + C_{33}^- + C_{34}^-\right] \\
+ \; & Yys\left[-2C_{11}^- + 2C_{12}^- - 3C_{21}^- + C_{22}^- + 2C_{23}^- - C_{31}^- + C_{34}^-\right] \\
+ \; & m_t^2\left[C_{11}(tt) + C_{12}(tt)\right] + m_t^2 C_0(tt) + 6C_{24}^- + 6C_{35}^- + 6C_{36}^-
\end{aligned}
$$

$$
\begin{aligned}
I_6 \;=\; & x_2 ys\left[-C_{32}^- - C_{33}^- + 2C_{34}^-\right] \\
+ \; & Xys\left[-C_{12}^- + C_{22}^- - 2C_{23}^- - C_{33}^- + C_{34}^-\right] \\
+ \; & Yys\left[C_{21}^- + C_{22}^- - 2C_{23}^- + C_{31}^- - 2C_{33}^- + C_{34}^-\right] \\
- \; & m_t^2\left[C_{11}(tt) - C_{12}(tt)\right] - m_t^2 C_0(tt) \\
- \; & \frac{1}{3} - 2C_{24}^- - 6C_{35}^- + 6C_{36}^-
\end{aligned}
$$

$$I_7 = 4Xys\left[C_{12}^- + C_{23}^-\right]$$

$$I_8 = 2Xys\left[-C_{12}^+ - 2C_{22}^+ - C_{23}^+ - 2C_{34}^+\right]$$

$$I_9 = x_2ys\left[C_{12}^+ + C_{23}^+ - C_{32}^+ + C_{34}^+\right]$$
$$+ Xys\left[C_{12}^+ + C_{22}^+ + C_{23}^+ + C_{34}^+\right]$$
$$+ Yys\left[-C_{11}^+ - C_{21}^+ + C_{22}^+ - C_{23}^+ - C_{33}^+ + C_{34}^+\right]$$
$$- m_t^2 C_{12}\left(tt\right) - \frac{1}{2} + 2C_{24}^+ + 2C_{36}^+$$

$$I_{10} = 4Xys\left[C_{12}^- + C_{23}^-\right] \tag{143}$$

When computing the $t-b$ contribution to the vertex we find that the ultraviolet divergence is m_t-independent. One should remember that the whole calculation is organized in the following way: if we denote the vertex schematically by V, then

$$V = \sum_f V_f(\text{massless}) + \left[V_t(\text{massive}) - V_t(\text{massless})\right]. \tag{144}$$

The sum in Eq.(144) combines with the $1/g^2$-term to cancel the ultraviolet divergent term while the rest is a subtracted term that is ultraviolet finite. When we consider the massive-massive case, there are 24 form-factors. Computing the divergent part we find

$$W_{\mu\alpha\beta}^{\gamma WW}\big|_{\text{UV}} = \frac{g^3 s_\theta}{16\,\pi^2}\left(Xys\right)^{1/2}\frac{1}{3}\frac{1}{\bar{\varepsilon}}\left[W_{\mu\alpha\beta}^1 - 2\,W_{\mu\alpha\beta}^3\right.$$
$$+ \left. 2\,W_{\mu\alpha\beta}^5 - 4\,W_{\mu\alpha\beta}^6 - 2\,W_{\mu\alpha\beta}^7 - 2\,W_{\mu\alpha\beta}^8\right], \tag{145}$$

where the operators are given in Eq.(87). However, we easily derive the following relation

$$4\,\mathcal{N}\,V_{\mu\alpha\beta}^0 = W_{\mu\alpha\beta}^1 - 2\,W_{\mu\alpha\beta}^3 + 2\,W_{\mu\alpha\beta}^5$$
$$- 4\,W_{\mu\alpha\beta}^6 - 2\,W_{\mu\alpha\beta}^7 - 2\,W_{\mu\alpha\beta}^8, \tag{146}$$

showing, again, factorization of the divergence into the lowest order. For a massive internal world there are also $\gamma W\phi$ and $\gamma\phi\phi$ vertices. One finds again factorization of ultraviolet divergences, for instance

$$W_{\mu\beta}^{\gamma W\phi} = i\,g^3 s_\theta\,\mathcal{W}_{\mu\beta}^{\gamma W\phi},$$
$$W_{\mu\beta}^{\gamma W\phi}\big|_{\text{UV}} = \frac{i\,g^3 s_\theta}{16\,\pi^2}\frac{m_t^2}{M}\frac{3}{2}\frac{1}{\bar{\varepsilon}}\,\delta_{\mu\beta}. \tag{147}$$

The corresponding lowest order vertex gives $i\,gs_\theta M\delta_{\mu\beta}$. The two combine into the following expression:

$$C_{\mu\beta} = \frac{M^2}{g^2}\,\delta_{\mu\beta} + M\,\mathcal{W}^{\gamma W\phi}_{\mu\beta}, \tag{148}$$

and

$$\begin{aligned}
C_{\mu\beta}\,|_{\mathrm{UV}} &= p_W\Big[\frac{1}{G_W} + \frac{1}{16\,\pi^2}\,\frac{f_W(p_W)}{p_W}\Big]_{\mathrm{UV}} + \frac{1}{16\,\pi^2}\,m_t^2\,\frac{3}{2}\,\frac{1}{\bar\varepsilon} \\
&= \frac{1}{16\,\pi^2}\Big[f_W(p_W)_{\mathrm{UV}} + \frac{3}{2}\,m_t^2\,\frac{1}{\bar\varepsilon}\Big] = 0. \tag{149}
\end{aligned}$$

Similar results hold for the other combinations, $\gamma\phi W$ and $\gamma\phi\phi$.

These observations, completing the argument shown in Eq.(140), prove that the amplitude is ultraviolet finite in the Fermion-Loop scheme.

As we have shown, the ε-terms do not cancel in Eq.(143). They only do for a complete massless generation, $l\,\nu_l\,u\,d$. Indeed, it is very easy to show that the ε-terms are proportional to the hyper-charge so that in the total they cancel, but they cancel only if we add the full content of the three fermionic generations while keeping all the fermions (including the top quark) at zero mass. With a massive top quark they do not cancel and give an m_t-dependent contribution.

8 Re-summations in the neutral sector.

Dyson re-summation of transitions in the neutral sector are also available and we will proceed to their construction. First, we define indices $i, j, \ldots = 1, 2$ with $1 \equiv Z$ and $2 \equiv \gamma$. The transitions are given by

$$S = \frac{g^2}{16\,\pi^2}\,\Sigma, \qquad \Sigma^{ij}_{\mu\nu} = \Sigma^{ij}_0\,\delta_{\mu\nu} + \Sigma^{ij}_1\,p_\mu p_\nu. \tag{150}$$

The re-summed propagators are obtained in the usual way, namely

$$\begin{aligned}
\bar\Delta^{ij}_{\mu\nu} &= \Delta^{ij}\,\delta_{\mu\nu+}\Delta^{il}\,S^{lk}_{\mu\nu}\,\Delta^{kj} + \ldots \\
&= \Delta^{il}\,\delta_{\mu\alpha}\,X^{lj}_{\alpha\nu}, \quad X = (1 - S\,\Delta)^{-1}. \tag{151}
\end{aligned}$$

It is convenient to write

$$(1 - S\,\Delta)_{\mu\nu} = K\,\delta_{\mu\nu} + H\,p_\mu p_\nu, \tag{152}$$

with K and H matrices defined by

$$K = \begin{pmatrix} 1 - S^{zz}_0\Delta_z & -S^{z\gamma}_0\Delta_\gamma \\ -S^{\gamma z}_0\Delta_z & 1 - S^{\gamma\gamma}_0\Delta_\gamma \end{pmatrix} \qquad H = -\begin{pmatrix} S^{zz}_1\Delta_z & S^{z\gamma}_1\Delta_\gamma \\ S^{\gamma z}_1\Delta_z & S^{\gamma\gamma}_1\Delta_\gamma \end{pmatrix} \tag{153}$$

Let $X = I\,\delta_{\mu\nu} + J\,p_\mu p_\nu$, then one can easily find a solution of the following form:

$$\begin{aligned}
I &= K^{-1}, \qquad J = -\left(K + p^2\,H\right)^{-1}\,H\,K^{-1}, \\
X_{\mu\nu} &= \Big[\delta_{\mu\nu} - \left(K + p^2\,H\right)^{-1}\,H\,p_\mu p_\nu\Big]\,K^{-1}. \tag{154}
\end{aligned}$$

39

The re-summed propagators are

$$\bar{\Delta}_{\mu\nu}^{ij} = \Delta^i \, X_{\mu\nu}^{ij}, \tag{155}$$

where we have found that

$$X_{\mu\nu} = K^{-1} \, P_{\mu\nu} + T^{-1} \, L_{\mu\nu}. \tag{156}$$

The Lorentz-tensors appearing in the previous equation are defined by

$$P_{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad L_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}, \tag{157}$$

the remaining matrices being expressed as

$$
\begin{aligned}
K^{ij} &= \delta^{ij} - S_0^{il} \, \Delta_{lj}, \quad H^{ij} = -S_1^{il} \, \Delta_{lj}, \\
T^{ij} &= K^{ij} + p^2 \, H^{ij} = \delta^{ij} - t^{il} \, \Delta_{lj}.
\end{aligned} \tag{158}
$$

These results can be simplified, as we have done in the charged sector, by using Ward identities for two-point functions, i.e.

$$
\begin{aligned}
i\, p^\mu \, S_{\mu\nu}^{\gamma\gamma} &= i\, t^{\gamma\gamma} \, p_\nu = 0, \\
i\, p^\mu \, S_{\mu\nu}^{\gamma Z} &= i\, t^{\gamma Z} \, p_\nu = 0, \\
i\, p^\mu \, S_{\mu\nu}^{Z\gamma} &= i\, t^{Z\gamma} \, p_\nu = 0,
\end{aligned} \tag{159}
$$

giving $t^{\gamma\gamma} = t^{\gamma Z} = t^{Z\gamma} = 0$. Therefore the matrix T^{-1} is diagonal, with entries

$$T_{zz}^{-1} = \frac{1}{1 - t^{zz} \, \Delta_z}, \qquad T_{\gamma\gamma}^{-1} = 1. \tag{160}$$

The re-summed photon propagator becomes

$$\bar{\Delta}_{\mu\nu}^{\gamma\gamma} = \left[p^2 - S_0^{\gamma\gamma} - \frac{\left(S_0^{\gamma z}\right)^2}{p^2 + M_0^2 - S_0^{zz}} \right] P_{\mu\nu} + \frac{1}{p^2} \, L_{\mu\nu}. \tag{161}$$

The re-summed photon propagator, in the 't Hooft-Feynman gauge, satisfies the same relation as the bare propagator,

$$p^\mu \, \Delta_{\mu\nu}^{\gamma\gamma} = p^\mu \, \bar{\Delta}_{\mu\nu}^{\gamma\gamma} = \frac{p_\nu}{p^2}. \tag{162}$$

The $Z - \gamma$ re-summed transition is

$$\bar{\Delta}_{\mu\nu}^{\gamma z} = \frac{S_0^{\gamma z}}{\left(p^2 + M_0^2 - S_0^{zz}\right), (p^2 - S_0^{\gamma\gamma}) - \left(S_0^{\gamma z}\right)^2} \, P_{\mu\nu}. \tag{163}$$

It is convenient to re-express the results of Eqs.(161–163) in terms of running couplings. We obtain

$$
\begin{aligned}
\bar{\Delta}_{\mu\nu}^{\gamma\gamma} &= \left\{ -\frac{e^2(p^2)}{e^2 \, p^2} + \frac{s_\theta^2}{c_\theta^2} \left[1 - \frac{s^2(p^2)}{s_\theta^2} \right]^2 G_z(p^2) \right\} P_{\mu\nu} + \frac{L_{\mu\nu}}{p^2}, \\
\bar{\Delta}_{\mu\nu}^{\gamma z} &= \frac{s_\theta}{c_\theta} \left[1 - \frac{s^2(p^2)}{s_\theta^2} \right] G_z(p^2) \, P_{\mu\nu}.
\end{aligned} \tag{164}
$$

40

The propagator-function G_z is given in Eq.(20). For fermionic loops there is no transition $\gamma - \phi^0$, while

$$S^\mu_{\phi z} = \frac{g^2}{16\,\pi^2 c_\theta^2}\,\Sigma_{\phi z}\,i p^\mu, \quad \Sigma_{\phi z} = -\frac{1}{2}\,\frac{m_t^2}{M}\,B_0\left(p^2; m_t, m_t\right). \tag{165}$$

In our conventions, we denote with a subscript ϕ the ϕ^0-field. Therefore, for non-zero m_t, we have to complete the diagonalization. Let us introduce indices $a, b, \ldots = 1, \ldots, 5$ with

$$S_{55} = S_{\phi\phi}, \quad S_{\mu 5} = S^\mu_{z\phi}, \quad S_{5\nu} = S^\nu_{\phi z}. \tag{166}$$

One should remember that the transitions S_{VV} have already been re-summed in $\bar{\Delta}^{zz}_{\mu\nu}$ etc, therefore the additional re-summation gives

$$\begin{aligned}
\tilde{\Delta}_{ab} &= \Delta_{ab} + \Delta_{ac}\,S_{cd}\,\Delta_{db} + \ldots = \Delta_{ac}\,X_{cb}, \\
X_{ab} &= (1 - S\,\Delta)^{-1}_{ab}.
\end{aligned} \tag{167}$$

In this equation one should use

$$\Delta_{\mu\nu} \equiv \bar{\Delta}^{zz}_{\mu\nu}, \qquad \Delta_{\phi\phi} \equiv \Delta_\phi. \tag{168}$$

The matrix to be inverted is

$$\begin{pmatrix} \delta_{\mu\nu} & +i\,S_{z\phi}\,\Delta_\phi\,p^\mu \\ -i\,S_{z\phi}\,p^\mu\,\bar{\Delta}^{zz}_{\mu\nu} & 1 - S_{\phi\phi}\,\Delta_\phi \end{pmatrix} \tag{169}$$

however, it is easily found that the following property holds:

$$p^\mu\,\bar{\Delta}^{zz}_{\mu\nu} = \Delta_z\,T^{-1}_{zz}\,p^\nu = \frac{1}{p^2 + M_0^2 - t^{zz}}\,p^\nu. \tag{170}$$

There are additional Ward identities in the neutral sector, corresponding to two-point functions, that give

$$i\,p^\mu\,S^{\mu\nu}_{zz} + M_0\,S_{\phi z}\,i\,p^\nu = (t^{zz} + M_0\,S_{\phi z})\,i\,p^\nu = 0, \tag{171}$$

$$\begin{aligned}
& i\,p_\mu\,S^{\mu\nu}_{zz}\,(-i\,p_\nu) + i\,p_\mu\,M_0\,(-S_{\phi z})\,i\,p^\mu \\
- \; & i\,p_\mu\,M_0\,(S_{\phi z})\,i\,p^\mu + M_0^2\,S_{\phi\phi} \\
= \; & p^2\,t^{zz} + 2\,p^2\,M_0\,S_{\phi z} + M_0^2\,S_{\phi\phi} = 0.
\end{aligned} \tag{172}$$

The solution to these conditions is

$$S_{\phi z} = -\frac{M_0}{p^2}\,S_{\phi\phi}, \qquad t^{zz} = \frac{M_0^2}{p^2}\,S_{\phi\phi}, \tag{173}$$

allowing to express everything in terms of the $\phi^0 - \phi^0$ transition. In this way our matrix becomes

$$\begin{pmatrix} \delta_{\mu\nu} & i\,b\,p_\mu \\ -i\,b'\,p_\nu & c \end{pmatrix} \tag{174}$$

with entries given by

$$b = -\frac{M_0}{p^2} S_{z\phi} \Delta_z, \qquad b' = -\frac{M_0}{p^2} \frac{S_{z\phi}}{p^2 + M_0^2 - t^{zz}},$$

$$c = 1 - S_{\phi\phi} \Delta_z, \tag{175}$$

where we have used the fact that in the 't Hooft-Feynman gauge $\Delta_z = \Delta_\phi = 1/(p^2 + M_0^2)$. After inversion of the matrix we obtain

$$\frac{1}{c - bb'p^2} \begin{pmatrix} (c - bb'p^2)\, \delta_{\mu\nu} + bb'\, p_\mu p_\nu & -i\, b\, p_\mu \\ +i\, b'\, p_\nu & 1 \end{pmatrix} \tag{176}$$

This result allows us to write down the remaining re-summed propagators. They are:

$$\tilde{\Delta}_{\mu\nu}^{zz} = \left[p^2 + M_0^2 - S_0^{zz} - \frac{(S_0^{z\gamma})^2}{p^2 - S_0^{\gamma\gamma}} \right]^{-1} P_{\mu\nu}$$

$$+ \frac{p^2 (p^2 + M_0^2 - S_{\phi\phi})}{(p^2 + M_0^2)^2 (p^2 - S_{\phi\phi})} L_{\mu\nu},$$

$$\tilde{\Delta}_{\phi\phi} = \frac{p^2 (p^2 + M_0^2) - M_0^2 S_{\phi\phi}}{(p^2 + M_0^2)^2 (p^2 - S_{\phi\phi})}$$

$$\tilde{\Delta}_{z\phi}^{\mu} = -\tilde{\Delta}_{\phi z}^{\mu} = i\, p^\mu M_0 \frac{S_{\phi\phi}}{(p^2 + M_0^2)^2 (p^2 - S_{\phi\phi})}. \tag{177}$$

8.1 The running Z mass.

With the results for transitions in the neutral sector, we can compute the amplitude for $e^+(p_+)\, e^-(p_-) \to f(q_-)\, \overline{f}(q_+)$. There are seven diagrams contributing, when we do not neglect the fermion masses. Accordingly, the amplitude is

$$M = g^2 Q_f s_\theta^2\, \gamma^\mu \otimes \gamma^\mu\, \bar{\Delta}_{\mu\nu}^{\gamma\gamma}$$

$$+ \frac{g^2 s_\theta}{2\, c_\theta} \gamma^\mu \otimes \gamma^\nu \left(I_f^{(3)} \gamma_+ - 2\, Q_f\, s_\theta^2 \right) \bar{\Delta}_{\mu\nu}^{\gamma z}$$

$$- \frac{g^2 Q_f\, s_\theta}{2\, c_\theta} \gamma^\mu \left(-\frac{1}{2} \gamma_+ + 2\, s_\theta^2 \right) \otimes \gamma^\nu\, \bar{\Delta}_{\mu\nu}^{z\gamma}$$

$$- \frac{g^2}{4\, c_\theta^2} \gamma^\mu \left(-\frac{1}{2} \gamma_+ + 2\, s_\theta^2 \right) \otimes \gamma^\nu \left(I_f^{(3)} \gamma_+ - 2\, Q_f\, s_\theta^2 \right) \tilde{\Delta}_{\mu\nu}^{zz}$$

$$- \frac{g^2}{2\, c_\theta} I_f^{(3)} \frac{m_f}{M} \gamma^\mu \left(-\frac{1}{2} \gamma_+ + 2\, s_\theta^2 \right) \otimes \gamma_5\, \bar{\Delta}_{\mu}^{z\phi}$$

$$+ \frac{g^2}{2\, c_\theta} \frac{m_e}{2\, M} \gamma_5 \otimes \gamma^\mu \left(I_f^{(3)} \gamma_+ - 2\, Q_f\, s_\theta^2 \right) \tilde{\Delta}_{\mu}^{\phi z}$$

$$+ \frac{g^2}{2} I_f^{(3)} \frac{m_e m_f}{M^2} \gamma_5 \otimes \gamma_5\, \tilde{\Delta}^{\phi\phi}, \tag{178}$$

where $I_f^{(3)}$ is the third component of isospin. We extract from the first four terms the part of the vector-vector propagators proportional to $\delta_{\mu\nu}$ and obtain the familiar $\gamma \oplus Z$ exchanges with running couplings. This part of the amplitude will be denoted by $M_{\delta\delta}$ and we write

$$M = M_{\delta\delta} + M_{\text{extra}}. \tag{179}$$

From the use of the Dirac equation we find

$$
\begin{aligned}
\slashed{p} \quad &\otimes \quad \slashed{p} = \slashed{p} \otimes \slashed{p}\,\gamma_+ = \slashed{p}\,\gamma_+ \otimes \slashed{p} = 0, \\
\slashed{p}\,\gamma_+ \quad &\otimes \quad \slashed{p}\,\gamma_+ = 4\,m_e m_f\,\gamma_5 \otimes \gamma_5, \\
\slashed{p}\,\gamma_+ \quad &\otimes \quad \gamma_5 = -2\,i\,m_e\,\gamma_5 \otimes \gamma_5, \qquad \gamma_5 \otimes \slashed{p}\,\gamma_+ = 2\,i\,m_f\,\gamma_5 \otimes \gamma_5. \tag{180}
\end{aligned}
$$

Therefore, collecting the various terms, we obtain the following expression

$$
\begin{aligned}
M_{\text{extra}} \quad = \quad & g^2\,m_e m_f\,I_f^{(3)}\,\gamma_5 \otimes \gamma_5 \left\{ \frac{1}{2\,c_\theta^2} \left[-G_z + \frac{p^2\left(p^2 + M_0^2 - S_{\phi\phi}\right)}{\left(p^2 + M_0^2\right)^2\left(p^2 - S_{\phi\phi}\right)} \right] \frac{1}{p^2} \right. \\
& \left. + \quad \frac{1}{c_\theta}\frac{M_0}{M}\frac{S_{\phi\phi}}{\left(p^2 + M_0^2\right)^2\left(p^2 - S_{\phi\phi}\right)} + \frac{1}{2\,M^2}\frac{p^2\left(p^2 + M_0^2 - \frac{M_0^2}{p^2}S_{\phi\phi}\right)}{\left(p^2 + M_0^2\right)^2\left(p^2 - S_{\phi\phi}\right)} \right\}.
\end{aligned}
\tag{181}
$$

Definition 2 *This result can be simplified considerably if we introduce a running Z-mass. We define*

$$M_{\text{extra}} = \frac{g^2}{2\,c_\theta^2}\,m_e m_f\,I_f^{(3)}\,\gamma_5 \otimes \gamma_5\,\frac{G_z}{M_0^2(p^2)}, \tag{182}$$

so that the entire amplitude can be written as the Born-like sum of γ and Z exchange, but with $g^2 \to g^2(p^2), s_\theta^2 \to s^2(p^2)$ etc and with an *effective Z* propagator given by

$$\Delta_{\mu\nu}^{ZZ,\text{eff}} = G_z(p^2)\left[\delta_{\mu\nu} + \frac{p_\mu p_\nu}{M_0^2(p^2)}\right]. \tag{183}$$

For a massless internal world we obtain a rather simple result:

$$
\begin{aligned}
\frac{1}{M_0^2(p^2)} \quad &= \quad \frac{1}{p^2}\left[\frac{1}{M_0^2\,G_z(p^2)} - 1\right] \\
&= \quad \frac{g^2(p_z)}{g^2(p^2)}\frac{c^2(p^2)}{c^2(p_z)}\frac{1}{p_z}.
\end{aligned}
\tag{184}
$$

which satisfies the relation $M_0^2(p_z) = p_z$.

As expected the Z and W running masses satisfy the same relation as the bare masses,

$$\frac{1}{M_0^2(p^2)} = \frac{c^2(p^2)}{M^2(p^2)}, \tag{185}$$

while complex poles satisfy

$$p_z = \frac{g^2(p_z)}{g^2(p_w)} \frac{p_w}{c^2(p_z)}.$$ (186)

The extension to non-zero m_t is straightforward.

9 Projection into the Imaginary Fermion-Loop.

All relevant Ward identities are linear in the vertex functions and the inverse propagators. As a consequence the real and imaginary parts fulfill the Ward identities separately. Therefore, a simplified minimal approach to incorporate the finite width while ensuring both U(1) and SU(2) gauge invariance consists in only taking into account the imaginary parts of fermionic corrections.

Having the full Fermion-Loop at our disposal we can project the result into this minimal version. The relevant formulas are listed below. First, the two-point functions. With $s = -p^2$ we write:

$$\mathrm{Im}B_0\left(p^2; 0, 0\right) = \pi\theta(s), \quad \mathrm{Im}B_0\left(p^2; 0, m\right) = \pi\left(1 - \frac{m^2}{s}\right)\theta\left(s - m^2\right),$$

$$\mathrm{Im}B_0\left(p^2; m, m\right) = \pi\left(1 - 4\frac{m^2}{s}\right)^{1/2}\theta\left(s - 4m^2\right),$$ (187)

Furthermore, when $p^2 = -s_p$ and s_p is a complex pole, the relevant functions become as follows:

$$B_0\left(-s_p; 0, 0\right) = \frac{1}{\bar{\varepsilon}} + 2 - \left[\ln\left(\frac{-s_p}{\mu^2}\right) - 2\,i\,\pi\right],$$

$$B_0\left(-s_p; 0, m\right) = \frac{1}{\bar{\varepsilon}} + 2 - \ln\frac{m^2}{\mu^2}\left(1 - \frac{m^2}{s_p}\right)\left[\ln\left(1 - \frac{s_p}{m^2}\right)\right.$$
$$\left. - 2\,i\,\pi\,\theta\left(M^2 - m^2\right)\right],$$

$$B_0\left(-s_p; m, m\right) = \frac{1}{\bar{\varepsilon}} + 2 - \ln\frac{m^2}{\mu^2} - \beta\left[\ln\frac{\beta + 1}{\beta - 1} - 2\,i\,\pi\,\theta\left(M^2 - 4\,m^2\right)\right],$$
(188)

where $M^2 = \mathrm{Re}\left(s_p\right)$ and $\beta^2 = 1 - 4\,m^2/s_p$. After scalarization the vertices are combinations of B_0-functions and of C_0-functions. The imaginary part of a C_0-function, in the general case, is not particularly simple but it is obtained from its complete expression [12]. An alternative way of deriving it is to remember that the imaginary part follows from the sum over all cuts of the scalar three-point function. In the generalization of the Fermion-Loop and in proving the Ward identity one should remember to sum over the three cuts, i.e., the imaginary part of the triangle VWW $(V = Z, \gamma)$ is the sum of all cuts over

intermediate states. Here we give one explicit example where we consider the function $C_0\left(p_1^2, p_2^2, P^2; m_1, m_2, m_3\right)$. The cuts give

$$
\begin{aligned}
C_{13} &= \int d^4q\, \theta\left(-q_0\right)\, \theta\left(q_0 + P_0\right)\, \frac{\delta\left(q^2 + m_1^2\right)\, \delta\left((q+P)^2 + m_3^2\right)}{(q+p_1)^2 + m_2^2}, \\
C_{23} &= \int d^4q\, \theta\left(q_0 + p_{10}\right)\, \theta\left(-q_0 - P_0\right)\, \frac{\delta\left((q+p_1)^2 + m_2^2\right)\, \delta\left((q+P)^2 + m_3^2\right)}{q^2 + m_1^2}, \\
C_{12} &= \int d^4q\, \theta\left(q_0\right)\, \theta\left(-q_0 - p_{10}\right)\, \frac{\delta\left(q^2 + m_1^2\right)\, \delta\left((q+p_1)^2 + m_2^2\right)}{(q+P)^2 + m_3^2}, \quad (189)
\end{aligned}
$$

where the subscript ij specify the cut: C_{ij} is the diagram where the cut propagators correspond to internal masses m_i, m_j. If we introduce the following quantities:

$$
\begin{aligned}
a^2 &= \frac{1}{4\,s}\, \lambda\left(-s, p_1^2, p_2^2\right), \\
b &= -\frac{1}{2}\frac{-s + p_1^2 - p_2^2}{\sqrt{s}}, \\
c &= \frac{1}{2\sqrt{s}}\left(m_3^2 - m_1^2 - s\right), \\
d^2 &= \frac{1}{4\,s}\, \lambda\left(s, m_1^2, m_3^2\right), \quad (190)
\end{aligned}
$$

with $P^2 = -s$ and λ the Kállen λ-function, then we obtain

$$
C_{13} = \frac{\pi}{4\,a\sqrt{s}}\, \theta\left(s - (m_1 + m_3)^2\right)\, \ln\frac{m_2^2 - m_1^2 + p_1^2 - 2\,cb + 2\,ad}{m_2^2 - m_1^2 + p_1^2 - 2\,cb - 2\,ad}, \quad (191)
$$

where the θ-function reflects the positivity of $\lambda(s, m_1^2, m_3^2)$ and the constraint $s \geq |\, m_1^2 - m_3^2\, |$. Similar results can be derived for the other two cuts. Note, however, that once the massive top contributions are included this scheme is in fact not much easier than the complete Fermion-Loop scheme.

10   Conclusions.

The Fermion-Loop scheme developed in [1] and refined in [2] makes the approximation of neglecting all masses for the incoming and outgoing fermions in the processes $e^+e^- \to n$ fermions.

In this paper we have given the construction of a complete, massive-massive, Fermion-Loop scheme, i.e., a scheme for incorporating the finite-width effects in the theoretical predictions for tree-level, LEP 2 and beyond, processes. Therefore, we have generalized the scheme to incorporate external non-conserved currents.

45

We work in the 't Hooft-Feynman gauge and create all relevant building blocks, namely the vector-vector, vector-scalar and scalar-scalar transitions of the theory, all of them one-loop re-summed. The loops, entering the scheme, contains fermions and, as done before in [2], we allow for a non-zero top quark mass inside loops. There is a very simple relation between re-summed transitions and running parameters, since Dyson re-summation is most easily expressed in terms of running coupling constants and running sinuses.

In our generalization, we have found particularly convenient to introduce additional running quantities. They are the running masses of the vector bosons, $M_0^2(p^2) = M^2(p^2)/c^2(p^2)$, formally connected to the location of the W and Z complex poles.

After introducing these running masses, it is relatively simple to prove that all $\mathcal{S}$-matrix elements of the theory assume a very simple structure. Coupling constants, sinuses and masses are promoted to running quantities and the $\mathcal{S}$-matrix elements retain their Born-like structure, with running parameters instead of bare ones, and vector-scalar or scalar-scalar transitions disappear if we employ unitary-gauge–like vector boson propagators where the masses appearing in the denominator of propagators are the running ones.

Renormalization of ultraviolet divergences has been easily extended to the massive-massive case by showing that all ultraviolet divergent parts of the one-loop vertices, $\gamma WW, \gamma W\phi, \gamma\phi W$ and $\gamma\phi\phi$ for instance, are proportional to the lowest order part. Therefore, the only combinations that appear are of the form $1/g^2 + VVV$ vertex or $M^2/g^2 + VV\phi$ vertex etc. All of them are, by construction, ultraviolet finite.

Equipped with our generalization of the Fermion-Loop scheme, we have been able to prove the fully massive U(1) Ward identity which guarantees that our treatment of the single-W processes is the correct one. As a by-product of the method, the cross-section for single-W production automatically evaluates all channels at the right scale, without having to use ad hoc re-scalings and avoiding the approximation of a unique scale for all terms contributing to the cross-section.

The generalization of the Fermion-Loop scheme goes beyond its, most obvious, application to single-W processes and allows for a gauge invariant treatment of all $e^+e^- \to n$ fermion processes with a correct evaluation of the relevant scales. Therefore, our scheme can be applied to several other processes like $e^+e^- \to Z\gamma^*$ and, in general to all $e^+e^- \to 6$ fermion processes, with the inclusion of top quarks.

References

[1] E. Argires et al., Phys. Lett. B358(1995)339.

[2] W. Beenakker, G. J. van Oldenborgh, A. Denner, S. Dittmaier, J. Hoogland, R. Kleiss, C. G. Papadopoulos and G. Passarino, hep-ph/9612260, Nucl. Phys. B500(1997)255.

[3] Y. Kurihara, D. Perret-Gallix and Y. Shimizu, Phys. Lett. B349(1995) 367.

[4] J. Hoogland and G. J. van Oldenborgh, Phys. Lett. B402(1197)379.

[5] G. Passarino, The Single W Production Case, hep-ph/9810416.

[6] E. Boos and M. Dubinin, hep-ph/9909214;
C. G. Papadopoulos, in Muenchen/Annecy/Hamburg 1992-93, e^+e^- collisions at 500-GeV, pt. C 199-205;
M. A. Doncheski, S. Godfrey and K. A. Peterson, hep-ph/9710299 ;
T. Tsukamoto, hep-ph/9611212, Acta Phys. Polon. B28(1997)695;
T. Tsukamoto and Y. Kurihara, hep-ph/9607353, Phys. Lett. B389(1996)162;
K. Huitu, J. Maalampi and M. Raidal, hep-ph/9602387;
C. G. Papadopoulos, hep-ph/9402208, Phys.Lett. B333(1994)202.

[7] G. Passarino, Comp. Phys. Comm. 97(1996)261.

[8] E. Accomando, A. Ballestrero and E. Maina, in preparation.

[9] A. Ballestrero, hep-ph/9911235.

[10] F. Berends, R. Kleiss and R. Pittau, Comp. Phys. Comm. 85(1995)437;
F. Caravaglios and M. Moretti, Phys. Lett. B308(1995)332;
G. Passarino, Comp. Phys. Comm. 97(1996)261;
J. Fujimoto et al., Comp. Phys. Comm. 100(1997)128;
D. G. Charlton et al., Comp. Phys. Comm. 99(1997)335;
E. Accomando and A. Ballestrero, Comp. Phys. Comm. 99(1997)270;
A. Pukhov et al, hep-ph/9908288;
S. Jadach et al., Comp. Phys. Comm. 119(1999)272.

[11] ALEPH Collaboration (R. Tanaka for the collaboration). CERN-OPEN-99-280, Jul 1999;
ALEPH Collaboration (R. Barate et al.). CERN-EP-99-086, Jun 1999, Phys.Lett. B462(1999)389;
R. Tanaka, X-LPNHE-98-08, In Vancouver 1998, High energy physics, vol. 1 515-519. hep-ex/9811039;
L3 Collaboration (M. Acciarri et al.). CERN-EP-98-099, Jun 1998, Phys. Lett. B436(1998)417;
L3 Collaboration (M. Acciarri et al.). CERN-PPE-97-028, Mar 1997, Phys. Lett. B403(1997)168.

[12] D. Bardin and G. Passarino, The Standard Model in the Making, Clarendon Press - Oxford, 1999.

[13] G. Passarino and M. Veltman, Nucl. Phys. B160(1979)151.